Chapter 5
The Production Process and Costs
Overview

I. Production Analysis
   – Total Product, Marginal Product, Average Product.
   – Isoquants.
   – Isocosts.
   – Cost Minimization

II. Cost Analysis
   – Total Cost, Variable Cost, Fixed Costs.
   – Cubic Cost Function.
   – Cost Relations.

III. Multi-Product Cost Functions
Production Analysis

- Production Function
  - $Q = F(K, L)$
    - $Q$ is quantity of output produced.
    - $K$ is capital input.
    - $L$ is labor input.
    - $F$ is a functional form relating the inputs to output.
  - The maximum amount of output that can be produced with $K$ units of capital and $L$ units of labor.

- Short-Run vs. Long-Run Decisions
- Fixed vs. Variable Inputs
Production Function Algebraic Forms

- Linear production function: inputs are perfect substitutes.
  \[ Q = F(K, L) = aK + bL \]
- Leontief production function: inputs are used in fixed proportions.
  \[ Q = F(K, L) = \min\{bK, cL\} \]
- Cobb-Douglas production function: inputs have a degree of substitutability.
  \[ Q = F(K, L) = K^a L^b \]
Productivity Measures: Total Product

- Total Product (TP): maximum output produced with given amounts of inputs.

- Example: Cobb-Douglas Production Function:
  \[ Q = F(K,L) = K^{0.5} L^{0.5} \]
  
  - K is fixed at 16 units.
  
- Short run Cobb-Douglas production function:
  \[ Q = (16)^{0.5} L^{0.5} = 4 L^{0.5} \]
  
  - Total Product when 100 units of labor are used?
  \[ Q = 4 (100)^{0.5} = 4(10) = 40 \text{ units} \]
Productivity Measures: Average Product of an Input

- Average Product of an Input: measure of output produced per unit of input.
  - Average Product of Labor: $\text{AP}_L = \frac{Q}{L}$.
  - Measures the output of an “average” worker.
  - Example: $Q = F(K,L) = K^{0.5}L^{0.5}$
    - If the inputs are $K = 16$ and $L = 16$, then the average product of labor is $\text{AP}_L = \frac{(16^{0.5})(16^{0.5})}{16} = 1$.
  - Average Product of Capital: $\text{AP}_K = \frac{Q}{K}$.
    - Measures the output of an “average” unit of capital.
    - Example: $Q = F(K,L) = K^{0.5}L^{0.5}$
      - If the inputs are $K = 16$ and $L = 16$, then the average product of capital is $\text{AP}_K = \frac{(16^{0.5})(16^{0.5})}{16} = 1$. 
Productivity Measures: Marginal Product of an Input

- Marginal Product on an Input: change in total output attributable to the last unit of an input.
  - Marginal Product of Labor: $MP_L = \frac{\Delta Q}{\Delta L}$
    • Measures the output produced by the last worker.
    • Slope of the short-run production function (with respect to labor).
  - Marginal Product of Capital: $MP_K = \frac{\Delta Q}{\Delta K}$
    • Measures the output produced by the last unit of capital.
    • When capital is allowed to vary in the short run, $MP_K$ is the slope of the production function (with respect to capital).
Increasing, Diminishing and Negative Marginal Returns

\[ Q = F(K, L) \]

Increasing Marginal Returns

Diminishing Marginal Returns

Negative Marginal Returns

Q

L

AP

MP
Guiding the Production Process

- Producing on the production function
  - Aligning incentives to induce maximum worker effort.

- Employing the right level of inputs
  - When labor or capital vary in the short run, to maximize profit a manager will hire:
    - labor until the value of marginal product of labor equals the wage: \( VMP_L = w \), where \( VMP_L = P \times MP_L \).
    - capital until the value of marginal product of capital equals the rental rate: \( VMP_K = r \), where \( VMP_K = P \times MP_K \).
Isoquant

- Illustrates the long-run combinations of inputs (K, L) that yield the producer the same level of output.
- The shape of an isoquant reflects the ease with which a producer can substitute among inputs while maintaining the same level of output.
Marginal Rate of Technical Substitution (MRTS)

- The rate at which two inputs are substituted while maintaining the same output level.

\[ MRTS_{KL} = \frac{MP_L}{MP_K} \]
Linear Isoquants

- Capital and labor are perfect substitutes
  - \( Q = aK + bL \)
  - \( \text{MRTS}_{KL} = b/a \)
  - Linear isoquants imply that inputs are substituted at a constant rate, independent of the input levels employed.
Leontief Isoquants

- Capital and labor are perfect complements.
- Capital and labor are used in fixed-proportions.
- \( Q = \min \{bK, cL\} \)
- Since capital and labor are consumed in fixed proportions there is no input substitution along isoquants (hence, no MRTS\(_{KL}\)).
Cobb-Douglas Isoquants

- Inputs are not perfectly substitutable.
- Diminishing marginal rate of technical substitution.
  - As less of one input is used in the production process, increasingly more of the other input must be employed to produce the same output level.

- \( Q = K^a L^b \)
- \( \text{MRTS}_{KL} = \frac{MP_L}{MP_K} \)
Isocost

- The combinations of inputs that produce a given level of output at the same cost: 
  \[ wL + rK = C \]
- Rearranging, 
  \[ K = \frac{(1/r)C}{w} - \frac{(w/r)L} \]
- For given input prices, isocosts farther from the origin are associated with higher costs.
- Changes in input prices change the slope of the isocost line.

New Isocost Line for a decrease in the wage (price of labor: \( w_0 > w_1 \)).

New Isocost Line associated with higher costs (\( C_0 < C_1 \)).
Cost Minimization

- Marginal product per dollar spent should be equal for all inputs:

\[ \frac{MP_L}{w} = \frac{MP_K}{r} \iff \frac{MP_L}{MP_K} = \frac{w}{r} \]

- But, this is just

\[ MRTS_{KL} = \frac{w}{r} \]
Cost Minimization

Slope of Isocost = Slope of Isoquant

Point of Cost Minimization
Optimal Input Substitution

- A firm initially produces $Q_0$ by employing the combination of inputs represented by point A at a cost of $C_0$.
- Suppose $w_0$ falls to $w_1$.
  - The isocost curve rotates counterclockwise; which represents the same cost level prior to the wage change.
  - To produce the same level of output, $Q_0$, the firm will produce on a lower isocost line ($C_1$) at a point B.
  - The slope of the new isocost line represents the lower wage relative to the rental rate of capital.
Cost Analysis

- **Types of Costs**
  - **Short-Run**
    - Fixed costs (FC)
    - Sunk costs
    - Short-run variable costs (VC)
    - Short-run total costs (TC)
  - **Long-Run**
    - All costs are variable
    - No fixed costs
Total and Variable Costs

C(Q): Minimum total cost of producing alternative levels of output:

\[ C(Q) = VC(Q) + FC \]

VC(Q): Costs that vary with output.

FC: Costs that do not vary with output.
Fixed and Sunk Costs

FC: Costs that do not change as output changes.

Sunk Cost: A cost that is forever lost after it has been paid.

Decision makers should ignore sunk costs to maximize profit or minimize losses.

\[ C(Q) = VC + FC \]
Some Definitions

Average Total Cost
\[ ATC = AV + AFC \]
\[ ATC = \frac{C(Q)}{Q} \]

Average Variable Cost
\[ AVC = \frac{VC(Q)}{Q} \]

Average Fixed Cost
\[ AFC = \frac{FC}{Q} \]

Marginal Cost
\[ MC = \frac{DC}{DQ} \]
Fixed Cost

\[ Q_0 \times (\text{ATC} - \text{AVC}) = Q_0 \times \text{AFC} = Q_0 \times (\frac{\text{FC}}{Q_0}) = \text{FC} \]
Variable Cost

\[ Q_0 \times AVC \]

\[ = Q_0 \times \left[ \frac{VC(Q_0)}{Q_0} \right] \]

\[ = VC(Q_0) \]

Minimum of AVC
Total Cost

\[ Q_0 \times \text{ATC} = Q_0 \times \left[ \frac{C(Q_0)}{Q_0} \right] \]

\[ = C(Q_0) \]

Minimum of ATC
Cubic Cost Function

- \( C(Q) = f + a \ Q + b \ Q^2 + cQ^3 \)
- Marginal Cost?
  - Memorize:
    \[ MC(Q) = a + 2bQ + 3cQ^2 \]
  - Calculus:
    \[ \frac{dC}{dQ} = a + 2bQ + 3cQ^2 \]
An Example

- Total Cost: \( C(Q) = 10 + Q + Q^2 \)
- Variable cost function:
  \[ VC(Q) = Q + Q^2 \]
- Variable cost of producing 2 units:
  \[ VC(2) = 2 + (2)^2 = 6 \]
- Fixed costs:
  \[ FC = 10 \]
- Marginal cost function:
  \[ MC(Q) = 1 + 2Q \]
- Marginal cost of producing 2 units:
  \[ MC(2) = 1 + 2(2) = 5 \]
Long-Run Average Costs

\[ \text{LRAC} \]

\[ Q^* \]

Economies of Scale

Diseconomies of Scale
Multi-Product Cost Function

- \( C(Q_1, Q_2) \): Cost of jointly producing two outputs.
- General function form:

\[
C(Q_1, Q_2) = f + aQ_1Q_2 + bQ_1^2 + cQ_2^2
\]
Economies of Scope

- \( C(Q_1, 0) + C(0, Q_2) > C(Q_1, Q_2) \).
  - It is cheaper to produce the two outputs jointly instead of separately.

- **Example:**
  - It is cheaper for Time-Warner to produce Internet connections and Instant Messaging services jointly than separately.
Cost Complementarity

- The marginal cost of producing good 1 declines as more of good two is produced:

\[ \frac{\Delta MC_1(Q_1, Q_2)}{\Delta Q_2} < 0. \]

- Example:
  - Cow hides and steaks.
Quadratic Multi-Product Cost Function

- \( C(Q_1, Q_2) = f + aQ_1Q_2 + (Q_1)^2 + (Q_2)^2 \)
- \( MC_1(Q_1, Q_2) = aQ_2 + 2Q_1 \)
- \( MC_2(Q_1, Q_2) = aQ_1 + 2Q_2 \)
- Cost complementarity: \( a < 0 \)
- Economies of scope: \( f > aQ_1Q_2 \)

\[
C(Q_1,0) + C(0,Q_2) = f + (Q_1)^2 + f + (Q_2)^2
\]

\[
C(Q_1,Q_2) = f + aQ_1Q_2 + (Q_1)^2 + (Q_2)^2
\]

\( f > aQ_1Q_2 \): Joint production is cheaper
A Numerical Example:

- \( C(Q_1, Q_2) = 90 - 2Q_1Q_2 + (Q_1)^2 + (Q_2)^2 \)
- Cost Complementarity?
  Yes, since \( a = -2 < 0 \)
  \( MC_1(Q_1, Q_2) = -2Q_2 + 2Q_1 \)
- Economies of Scope?
  Yes, since \( 90 > -2Q_1Q_2 \)
Conclusion

- To maximize profits (minimize costs) managers must use inputs such that the value of marginal of each input reflects price the firm must pay to employ the input.
- The optimal mix of inputs is achieved when the $MRTS_{KL} = (w/r)$.
- Cost functions are the foundation for helping to determine profit-maximizing behavior in future chapters.