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A simplified algebraic method for system of linear inequalities with LP applications☆

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ABSTRACT

In this pedagogical note we present an improved method to solve and analyze linear programming (LP) problems. The method depends on solving a system of equations and is free of any slack, surplus or artificial variables. The proposed method eliminates the need to manipulate linear inequalities to introduce additional variables and works only within the original decision variables space. We present applications of the method to handle linear optimization with varying objective function. The proposed method is easy to implement and enhances understanding of the simplex method and LP solvers transparent. We believe it is a useful alternative approach to present LP in the class room during the first few hours of introducing the subject.

1. Introduction

Since World-War II, linear optimization has been used to solve small and large problems in almost all business disciplines. Numerous applications of linear programming (LP) can be found in today's competitive business environment [1–6]. LP is a problem-solving approach to help managers make decisions. The graphical method of solving LP problems provides a clear illustration of the feasible and non-feasible regions, as well as, vertices. However, the graphical method has limitations in its applicability to solve LP problems having at most two decision variables.

The simplex method is essentially a combinatorial method and a combinatorial linear algebra over an order field. By definition, the number of basic solutions (BSs) associated with the solution space of $AX = b$ is limited by $n!/[m!(n-m)!]$, where A is an $m \times n$ matrix and b is $m \times 1$ vector. Application of algebraic simplex method of LP suggests finding an optimal solution to an LP by enumerating all possible BSs. The optimum is associated with the basic feasible solution (BFS) yielding the largest (smallest) objective value for a maximization (minimization) problem.

Solving LP problems in which some constraints are in ($\geqslant$) or ($=$) form with non-negative right-hand side (RHS) has raised difficulties. One version of the simplex, known as the two-phase method, introduces an artificial objective function, which is the sum of artificial variables. The other version adds the penalty terms, which are the sum of artificial variables with very large positive coefficients. The latter approach is known as the Big-M method [7]. Understanding the intuitive notion of artificial variables may require a greater mathematical sophistication from managers and the simplex methods have to iterate through many infeasible vertices to reach an initial feasible vertex [8]. Using the dual simplex method has its own difficulties. For example, when some coefficients in the objective function are not dual feasible, one must introduce an artificial constraint. Handling equality constraints by the dual simplex method is tedious because of introduction of two new variables for each equality constraint: one extraneous slack and one surplus.

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Also one may not be able to remove some equality constraints by elimination at the outset, as this may violate the non-negativity condition of some variables. In addition, degeneracy which may cause cycling may occur in both simplex and the dual simplex methods iterations [9].

In this note, we propose a new solution algorithm which is free from any extraneous surplus/slack variables, artificial variables, artificial objective functions, or artificial constraints. The algorithm initially concentrates on locating BSs by solving selected squared subsystems of equations with size dependent on the number of decision variables and constraints. Then, a feasibility test is performed on the obtained solution to be retained for further considerations. Since each iteration in fact is solving a system of equations, the algebraic approach presented in this paper acts as a bridge between the graphical and the simplex method.

The Improved Algebraic Method (IAM) proposed in this note works directly on the decision variable space in that no new variables are introduced regardless of the restriction on the signs of the variables. The method reduces considerably the computational complexity because it does not include any slack/surplus variables and enhances understanding of the simplex transparent. It is efficient, easy to understand, and can be applied to much larger set of linear inequalities than feasible by the standard algebraic approach. It is easy to explain the technique to students while introducing the topic. The advantage of the method is for small example and is not practical for real-life large LP. We present several numerical examples to illustrate the proposed method and its applications.

2. The LP problem

Problem P: Max (or Min) cX

Subject to:

$$\begin{aligned} &AX \leqslant a,\\ &BX \geqslant b,\\ &DX = d,\\ &X_i \geqslant 0, i = 1, \ldots, j,\\ &X_i \leqslant 0, i = j+1, \ldots, k,\\ &X_i \text{ unrestricted in sign,}\\ &\quad i = k+1, \ldots, n, \end{aligned}$$

where matrices A, B, and D have p, q, and r rows, respectively, with n columns and vectors c, a, b, and d have appropriate dimensions. Therefore, there are $m=(p+q+r+k)$ constraints and n decision variables. It is assumed that $m \geqslant n$. Note that the main constraints have been separated into three subgroups. Without loss of generality we assume that all RHS elements, a, b, and d are non-negative. We do not deal with trivial cases, such as where $A=B=D=0$ (no constraints), or $a=b=d=0$ (all boundaries pass through the origin point).

Our purpose here is to systematically develop and present an IAM to have the following useful features:

- solve the system of inequalities in an efficient manner;
- solve an LP with varying objective function;
- provide tight bounds on the objective function cX both maximization and minimization problems subject to the set of constraints;
- find a set of solutions to achieve a desirable value for the objective function cX, i.e., solving the goal-seeking problems.

3. Solving a system of linear inequalities

The solution to an LP problem is strictly based on the theory and solution of system of linear inequalities (SLIs) [1]. The BSs to a linear program are the solutions to the systems of equations consisting of constraints at binding position. Not all BSs satisfy all the problem constraints. Those that do meet all the constraint restrictions are called the basic feasible solutions. The BFSs correspond precisely to the vertices of the feasible region.

Definition 1. A solution to any system of equations is called a basic solution. Those BS which are feasible are called basic feasible solutions.

The main result of LP: The optimal solution of a bounded LP always occurs at a BFS, i.e., one of the vertices of the feasible region.

The importance of this fundamental theorem is that it reduces the LP problem to a "combinatorial" problem of determining which constraints out of many should be tight (binding) by the optimal solution.

4. The ordinary simplex algebraic method

The ordinary simplex algebraic method is a complete implicit enumerating algorithm to solve LP problems with bounded solutions. It converts all inequality constraints into equality constraints to obtain a system of equations by introducing slack/surplus variables, converts all non-restricted (in sign) variables to satisfy the required non-negativity conditions by substituting the difference of two new variables, and finally solves all of its square subsystems of equations. This conversion of an LP problem into a pure algebraic version ignores the original space of the decision variables and treats all variables alike throughout the process thus increasing the dimensionality and complexity.

Assuming an LP problem has a bounded solution; the ordinary simplex algebraic method proceeds as follows:

1. *Construction of the boundaries of the constraints set*: Transform all inequalities (except the restricted condition on each variable, if any) to equalities by adding or subtracting slack or surplus variables.
2. *Finding all vertices*: If the number of variables (including slack and surplus) is more than the number of equations, then set the following number of variables to zero: [(number of variables including slack and surplus)−(number of equations)]. After setting these many variables to zero, find the other variables by solving the resulting squared system of equations.
3. *Check for feasibility*: All slack and surplus variables must be non-negative and non-negativity condition on each variable should be satisfied. Determine all BFSs, vertices of the feasible region.

4. *Selecting the optimal corner point*: Among all BFSs, find the optimal one (if any) by evaluating the objective function. There might be multiple optimal solutions.

5. The IAM

We present an IAM of solving an SLIs that does not require the formulation of an auxiliary LP problem and solution algorithms such as Simplex. We provide a simple methodology to extend the solution of SLI of one or two dimensions to systems of higher dimensions. We are interested in finding the vertices of the feasible region of Problem P, expressed as a system of linear equalities and inequalities.

$$AX \leqslant a,$$
$$BX \geqslant b,$$
$$DX = d,$$

where some $X_i \geqslant 0$, some $X_i \leqslant 0$, and some X_i are unrestricted in sign. Matrices A, B, and D as well as vectors a, b, and d have appropriate dimensions. For the sake of convenience, we refer to this general system of equalities and inequalities as a "system" and its feasible region set as S. Therefore, the optimization problem can be expressed as

Problem P: Max (or min) $f(X)$
subject to: $X \in S$

If the objective function is nonlinear, the optimal solution to Problem P may be, in addition to the vertices, one of the many stationary points on the interior, faces, or edges of the feasible region. The interior of the bounded feasible region is defined by the convex set of all vertices obtained. Other relevant domains, such as faces, edges, etc., of the feasible region are defined by appropriate subsets of these vertices.

6. Steps of the IAM

Step 1: Convert all inequalities into equalities (including any variable restricted constraints).

Step 2: Calculate the difference between the number of variables n and the number of equations m.

Step 3: Determine solution to all square system of equations. The maximum number of systems of equation to be solved is: $m!/[n!(m-n)!]$.

Step 4: Check feasibility of each solution obtained in Step 3 by using the constraints of all other equations.

The coordinates of vertices are the BFSs of the systems of equations obtained by setting some of the constraints at binding (i.e., equality) position. For a bounded feasible region, the number of vertices is at most $m!/[n!(m-n)!]$ where m is the number of constraints and n is the number of variables. Therefore, a BS is obtained by taking any set of n equations and solving them simultaneously. By plugging this BS in the constraints of other equations, one can check for feasibility of the BS. If it is feasible, then this solution is a BFS that provides the coordinates of a corner point of the feasible region.

7. Numerical example

We provide an example to explain the IAM and develop parametric representation of the feasible region for the given SLI.

Example 1. Consider the following bounded feasible region:

$X_1 + X_2 + X_3 \leqslant 10$ Eq 1

$3X_1 + X_3 \leqslant 24$ Eq 2

$X_1 \geqslant 0$ Eq 3

$X_2 \geqslant 0$ Eq 4

$X_3 \geqslant 0$ Eq 5

There will be five equations and three variables yielding 10 possible combinations. The IAM provides six vertices for the feasible region as follows:

X_1	X_2	X_3	Feasible?	Binding equations
10	0	0	No	1, 4, and 5
8	0	0	Yes	2, 4, and 5
8	2	0	Yes	1, 2, and 5
0	10	0	Yes	1, 3, and 5
0	0	24	No	2, 3, and 4
0	0	10	Yes	1, 3, and 4
0	10	0	No	2, 3, and 5
7	0	3	Yes	1, 2, and 4
0	−14	24	No	1, 2, and 3
0	0	0	Yes	3, 4, and 5

Therefore, the six vertices are

$X_1 = 8$	$X_1 = 8$	$X_1 = 0$	$X_1 = 0$	$X_1 = 7$	$X_1 = 0$
$X_2 = 0$	$X_2 = 2$	$X_2 = 10$	$X_2 = 0$	$X_2 = 0$	$X_2 = 0$
$X_3 = 0$	$X_3 = 0$	$X_3 = 0$	$X_3 = 10$	$X_3 = 3$	$X_3 = 0$

Using the parameters $\lambda_1, \ldots, \lambda_6$ for the six vertices, we obtain the following parametric representation of the feasible region:

$$X_1 = 8\lambda_1 + 8\lambda_2 + 7\lambda_5$$

$$X_2 = 2\lambda_2 + 10\lambda_3$$

$$X_3 = 10\lambda_4 + 3\lambda_5$$

for all parameters $\lambda_1, \ldots, \lambda_6$ such that each $\lambda_i \geqslant 0$ and $\sum \lambda_i = 1$.

8. Applications to linear programs

The parametric representation of an SLI obtained from the IAM can be used to solve linear programs as illustrated by a numerical example in this section. The following example is an extension of the example presented previously.

Example 2.

$$\begin{aligned} \text{Max} \quad & 4X_1 + 2X_2 + 3X_3 \\ \text{subject to:} \quad & X_1 + X_2 + X_3 \leqslant 10 \\ & 3X_1 + X_3 \leqslant 24 \\ & X_1 \geqslant 0 \\ & X_2 \geqslant 0 \\ & X_3 \geqslant 0 \end{aligned}$$

As demonstrated in Example 1, the parametric representation of the feasible region with $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geqslant 0$ and $\sum \lambda_i = 1$ is as follows:

$$X_1 = 8\lambda_1 + 8\lambda_2 + 7\lambda_5$$

$$X_2 = 2\lambda_2 + 10\lambda_3$$

$$X_3 = 10\lambda_4 + 3\lambda_5$$

Substituting the parametric version of the feasible region into the objective function, we obtain:

$$f(\lambda) = 4X_1 + 2X_2 + 3X_3 = 32\lambda_1 + 36\lambda_2 + 20\lambda_3 + 30\lambda_4 + 37\lambda_5 \qquad (1)$$

The optimal solution occurs when $\lambda_5 = 1$ and all other λ_i's are set to 0, with a maximum value of 37. The optimal solution is ($X_1 = 7$, $X_2 = 0$, $X_3 = 3$), one of the vertices.

Proposition 2. *The maximum (minimum) points of an LP with a bounded feasible region correspond to the maximization (minimization) of the parametric objective function $f(\lambda)$.*

Let the terms with the largest (smallest) coefficients in $f(\lambda)$ be denoted by λ_L and λ_S, respectively. Since $f(\lambda)$ is a (linear) convex combination of its coefficients, the optimal solution of $f(\lambda)$ is obtained by setting λ_L or λ_S equal to 1 and all other $\lambda_i = 0$.

Lemma 3. *The maximum and minimum points of an LP with a bounded feasible region correspond to $\lambda_L = 1$ and $\lambda_S = 1$, respectively.*

If a polytope has a finite number of vertices, then this result suggests that the optimal solution to an LP problem can be found by enumerating all BFSs found by the IAM. The optimum is associated with the BFS yielding the largest or smallest objective value, assuming the problem is of maximization or minimization type, respectively.

8.1. Computing slack and surplus

Given that the RHS of a constraint is non-negative, the slack is the leftover amount of a resource ($\leqslant$) constraint, and surplus is the access over a requirement ($\geqslant$) constraint. These quantities represent the absolute values of the difference between the RHS value and the LHS (left-hand-side) evaluated at an optimal point. Having obtained an optimal solution, one can compute the slack and surplus for each constraint at optimality. Equality constraints are always binding with zero slack/surplus.

Since this numerical example is a three-dimensional LP, one expects (at least) to have three binding constraints. The binding constraints at optimality are Eqs. (1), (2), and (4):

$$X_1 + X_2 + X_3 = 10$$

$$3X_1 + X_3 = 24$$

$$X_2 = 0$$

The surplus values for the two non-binding constraints are $S_3 = 7$ and $S_5 = 3$.

8.2. Computation of shadow prices

By definition, the shadow price for a non-binding constraint is always zero. To compute the shadow price of binding constraints, excluding any non-negativity condition (in this example, $X_2 = 0$), one must first solve the following RHS parametric system of equations by plugging $X_2 = 0$,

$$X_1 + X_3 = 10 + R_1$$

$$3X_1 + X_3 = 24 + R_2$$

By setting all inequalities in binding positions and solving the system of equations, we get the following parametric solution:

$$X_1 = 7 - 0.5R_1 + 0.5R_2$$

$$X_2 = 0$$

$$X_3 = 3 + 1.5R_1 - 0.5R_2$$

The solution can be verified by substitution. For larger problem one may use the JavaScript available using the following link: http://home.ubalt.edu/ntsbarsh/Business-stat/otherapplets/PaRHSSyEqu.htm

Plugging the parametric solution into objective function, we have:

$$4X_1 + 2X_2 + 3X_3 = 37 + 2.5R_1 + 0.5R_2$$

The shadow price is a derivative of the parametric optimal function, i.e., $U_1 = 2.5$ and $U_2 = 0.5$, for two resources with RHS equal to 10 and 24, respectively. These shadow prices are the solution to the following corresponding dual problem:

$$\begin{aligned} \text{Min} \quad & 10U_1 + 24U_2 \\ \text{subject to:} \quad & U_1 + 3U_2 \geqslant 4 \\ & U_1 \geqslant 2 \\ & U_1 + U_2 \geqslant 3 \\ & U_1 \geqslant 0 \\ & U_2 \geqslant 0 \end{aligned}$$

The dual optimal value of 37 is equal to the optimal value of the primal problem, as expected.

8.3. Sensitivity analysis

Notice that the parametric objective function $37 + 2.5R_1 + 0.5R_2$ is valid when the parametric solution satisfies all unused constraints at the optimal solution. In the current numerical example, unused constraints are

$X_1 \geqslant 0$ and $X_3 \geqslant 0$

These produce the following largest set of sensitivity region for the RHS of all the constraints simultaneously:

$$7 - 0.5R_1 + 0.5R_2 \geqslant 0 \quad \text{and}$$
$$3 + 1.5R_1 - 0.5R_2 \geqslant 0 \tag{2}$$

8.3.1. Sensitivity ranges for the RHS of the constraints

The inequality set (2) can be used to find ranges for the RHS values of the constraints. The range for the RHS of the first constraint (RHS_1) can be obtained by setting $R_2=0$ in the inequalities set (2). This implies that $R_1 \leqslant 14$ and $R_1 \geqslant -2$. Therefore, the allowable increase and decrease in the original value 10 for RHS_1 are 14 and 2, respectively, i.e. ($8 \leqslant b_1 \leqslant 24$).

Similarly, the range for the RHS of the second constraint (RHS_2) can be obtained by setting $R_1 = 0$ in the inequality set (2). This implies that $R_2 \geqslant -14$ and $R_2 \leqslant 6$.

Therefore the allowable increase and decrease in the original value 24 for RHS_2 are 6 and 14, respectively, i.e. ($10 \leqslant b_2 \leqslant 30$).

8.3.2. Sensitivity ranges for the objective function coefficients

To find the ranges for the objective function coefficients, one may use the RHS parametric version of the dual problem; that is:

$$\begin{aligned} \text{Min} \quad & 10U_1 + 24U_2 \text{ RHS} \\ \text{subject to:} \quad & U_1 + 3U_2 \geqslant 4 + C_1 \\ & U_1 \geqslant 2 + C_2 \\ & U_1 + U_2 \geqslant 3 + C_3 \\ & U_1 \geqslant 0 \\ & U_2 \geqslant 0 \end{aligned}$$

As calculated earlier, the shadow prices are $U_1 = 2.5$ and $U_2 = 0.5$, which are the optimal solution to the dual. The first and the third constraints are binding. The parametric presentations of the RHS of these binding constraints are as follows:

$$U_1 + 3U_2 = 4 + C_1$$
$$U_1 + U_2 = 3 + C_3$$

Solving these equations we get the following parametric solution:

$$U_1 = 2.5 - 0.5C_1 + 1.5C_3$$
$$U_2 = 0.5 + 0.5C_1 - 0.5C_3$$

The parametric objective function is $37 + 7C_1 + 3C_3$ with optimal value of 37 for the nominal problem. Again, this parametric optimal solution is subject to satisfying the unused constraints, namely,

$U_1 \geqslant 2 + C_2$ and $U_2 \geqslant 0$

These produce the following largest sensitivity region for the objective function coefficients simultaneously:

$$0.5C_1 - C_2 + 1.5C_3 \geqslant -0.5 \quad \text{and}$$
$$0.5 + 0.5C_1 - 0.5C_3 \geqslant 0 \tag{3}$$

The sensitivity range of the coefficient of first decision variable X_1, currently at 4, can be found by setting $C_3 = 0$ and $C_2 = 0$ in the inequality set (3) yielding $C_1 \leqslant 1$ and $C_1 \geqslant -1$. Therefore, 1 is the allowable increase or decrease for the coefficient of X_1, i.e. ($3 \leqslant C_1 \leqslant 5$).

The sensitivity range of the coefficient of second decision variable X_2, currently at 2, can be found by setting $C_1 = 0$ and $C_3 = 0$, in the inequality set (3) yielding $C_2 \leqslant 0.5$. Therefore, 0.5 is the allowable increase in X_2 with no limit in decreasing the coefficient of X_1, i.e. ($C_2 \leqslant 2.5$).

The range for the coefficient of X_3 can be found by setting $C_1 = 0$ and $C_2 = 0$ in the inequality set (3) which yields $C_3 \leqslant 1$ and $C_3 \geqslant -\frac{1}{3}$. Therefore, the allowable increase and decrease for the coefficient of X_3 are 1 and $\frac{1}{3}$, respectively, i.e. ($\frac{8}{3} \leqslant C_3 \leqslant 4$).

9. Solving LP problem with changing objective function

The business environment is often unpredictable and uncertain because of factors such as economic changes, local government regulations, or dependence on subcontractors and vendors. Therefore, managers often find themselves in a dynamic, unsettled environment where even short range plans must be constantly reassessed and the objective function incrementally adjusted.

As discussed in the previous section, one can easily obtain a parametric representation, $f(\lambda)$, of a given objective function to determine the optimum value, whether maximum or minimum. Further, bounds on the range of the objective function value can be obtained. It can be easily seen that a practitioner does not have to entirely resolve the problem if the objective in the numerical example presented above is changed to Min (instead of Max). Looking at the parametric representation $f(\lambda)$ in Eq. (1), it is clear that the minimum value of 20 occurs at $\lambda_3 = 1$ that presents point ($X_1 = 0$, $X_2 = 10$, $X_3 = 0$). The objective function value is bounded by a minimum of 20 to a maximum of 37.

The parametric representation of the feasible region of an SLI is useful in solving the corresponding LP with varying objective. Consider the LP of Example 2 again. Now consider a slight change in the objective function. For example, if we decrease the third coefficient to 2.5, we have a new LP problem:

$$\begin{aligned} \text{Max} \quad & 4X_1 + 2X_2 + 2.5X_3 \\ \text{subject to:} \quad & X_1 + X_2 + X_3 \leqslant 10 \\ & 3X_1 + X_3 \leqslant 24 \\ & X_1 \geqslant 0 \\ & X_2 \geqslant 0 \\ & X_3 \geqslant 0 \end{aligned}$$

The new parametric objective function is

$$\begin{aligned} f(\lambda) &= 4X_1 + 2X_2 + 2.5X_3 \\ &= 32\lambda_1 + 36\lambda_2 + 20\lambda_3 + 25\lambda_4 + 35.5\lambda_5 \end{aligned} \tag{4}$$

Clearly, the optimal solution occurs when $\lambda_2 = 1$, and all other λ's are set to 0, with the maximum value of 36. Therefore, the optimal solution is $X_1 = 8$, $X_2 = 2$, $X_3 = 0$. That is, decreasing the third coefficient in the objective function by a slight amount changes the optimal solution significantly.

10. Extension to too-few constraints

As discussed earlier, the graphical method is limited in solving LP problems having one or two decision variables. One needs to determine all intersection points (vertices) and examine which one among all feasible vertices provides the optimal solution. The graphical method and the discussion so far assume that $n \leqslant m$. The IAM is designed to solve multi-dimensional LP problem $(n > m)$ regardless the number of constraints provided. In the following we discuss two such cases for illustration.

Example 3 $(n > m)$. Consider the feasible region of an SLI with m constraints and n decision variables. Suppose $n > m$ then every BS has at most m non-zero decision variables, i.e., at least $n - m$ decision variables must have zero value.

$$\begin{aligned} \text{Max (or Min)} \quad & 20X_1 + 30X_2 + 10X_3 + 40X_4 \\ \text{subject to:} \quad & X_1 + X_2 = 200 \\ & X_3 + X_4 = 100 \\ & X_1 + X_3 \geqslant 150 \end{aligned}$$

This LP problem cannot be solved by the graphical method. However, the algebraic method has no limitation on the LP dimension. One can compute all BSs by setting all constraints at binding positions:

$$X_1 + X_2 = 200$$

$$X_3 + X_4 = 100$$

$$X_1 + X_3 = 150$$

There are $n = 4$ decision variables and $m = 3$ constraints. Since $n > m$, at least $n - m (= 1)$ decision variable must be zero. Following are the possible BSs:

X_1	X_2	X_3	X_4	Feasible?	Objective value
0	200	150	−50	Yes	5500
200	0	−50	150	Yes	9500
150	50	0	100	Yes	8500
50	150	100	0	Yes	6500

Thus, from the above table, we obtain the following bounds on the objective function over its feasible region:

$$5500 \leqslant 20X_1 + 30X_2 + 10X_3 + 40X_4 \leqslant 9500 \qquad (5)$$

Result 4. Given a bounded LP with m constraints (excluding any sign constraints such as non-negativity conditions) and n decision variables, if $n > m$, then at most m decision variables have positive value at the optimal solution and the rest $(n - m)$ of the decision variables must be set at zero level. This result holds if the problem has a unique bounded optimal solution.

Remark 5. When adding to Example 3 the constraint $X_2 + X_4 \geqslant 150$ it mimics an LP formulation of the transportation problem with two origins (O_1 and O_2) and two destinations (D_1 and D_2) with the following supply and demand together with the unit transportation costs:

The Cost Matrix

	D_1	D_2	Supply
O_1	20	30	200
O_2	10	40	100
Demand	150	150	300

Notice that this transportation problem is a balanced one, i.e., total supply = total demand. Therefore, at least one of the constraints is redundant, this implies any one of the constraint can be deleted. Moreover, since all X_i values are supposed to be non-negative, a look at the table of BSs reveals that there are only two BFSs and the minimum optimum value is 6500.

11. Goal-seeking problems

Although some decision-makers would prefer the optimal, in most practical situations, however, a decision-maker aims at satisfying or making incremental changes rather than optimizing. This is so, because the human mind has a bounded rationality and hence cannot comprehend all alternatives. In many business applications, a manager may wish to achieve a specific achievable goal, while satisfying the constraints of the model. In the incremental approach to decision-making; the manager takes only small steps, or incremental moves, away from the existing system. This is usually accomplished by a "local search" to find a "good enough" solution. This problem is referred to as the "satisfying problem", "feasibility problem", or the "goal-seeking" problem. The user does not particularly want to optimize anything so there is no reason to define an objective function. The aim is to achieve a global improvement to a level that is good enough, given current information and resources.

To convert the goal-seeking problem to an optimization problem, one must first add the goal to the constraint set by creating a dummy objective function. It could be a linear combination of the subset of decision variables. By maximizing this objective function, one can get a feasible solution (if one exists). By minimizing it, one might get another solution (usually on the other "side" of the feasible region).

The proposed method with parametric representation provides an efficient approach that deals precisely with problems of constraint satisfaction without necessarily having an objective function.

Example 5. Consider the following goal-seeking problem:

$$\begin{aligned} \text{Goal } f(X): \quad & -X_1 + 2X_2 = 4 \\ \text{subject to:} \quad & X_1 + X_2 \geqslant 2 \\ & -X_1 + X_2 \geqslant 1 \\ & X_1 \geqslant 0 \\ & X_2 \geqslant 0 \\ & X_2 \leqslant 3 \end{aligned}$$

Adding this goal to the constraint set and using the IAM, two BFSs are obtained easily because the equality constraint must be present in all squared system of equations. The vertices of the feasible region are

$$X_1 = 0, \quad X_1 = 2$$
$$X_2 = 2, \quad X_2 = 3$$

Using the parametric representation of the goal parameters λ_1 and λ_2 for the first and the second vertices, respectively, we get:

$$X_1 = 2\lambda_2$$
$$X_2 = 2\lambda_1 + 3\lambda_2$$

for all parameters λ_1 and λ_2 such that $\lambda_1, \lambda_2 \geqslant 0$ and $\lambda_1+\lambda_2 = 1$. By substituting suitable values for these two parameters one can generate any strategy to achieve a desirable goal.

To verify, the parametric goal is

$$f(\lambda) = -X_1+2X_2 = 4\lambda_1+4\lambda_2 = 4(\lambda_1+\lambda_2) = 4, \quad \text{as expected.}$$

12. Conclusions

In this paper we present an improved method of solving a system of linear inequalities (SLI) that does not require formulation of an auxiliary (much larger) LP problem to be solved by algorithms such as the standard algebraic simplex method. We provide a simple methodology to extend the solution of SLI from two dimensions to systems of higher dimensions. Under the proposed method, the elaborate fundamental theorem of simplex method falls out as a by-product. Moreover, it can also be used to fill the gap between the graphical method of solving LP problems and the simplex method when teaching LP.

A business environment is dynamic. A problem solution is valid in a limited time window only and is subject to revision in the next time window. Generally, a constraint set is less subject to change as compared to the objective function. For example, in production and transportation problems, the capacity constraints may remain rather stable over a period of time. On the other hand, profit coefficients of the objective function are inversely related to the price, which may fluctuate, being determined by the market conditions and competition. The proposed method can be used to optimize LP problems with varying objective function. Given a system of linear equalities and/or inequalities, the method provides all vertices of the feasible region. A parametric representation of the feasible region as a convex combination of the vertices is developed.

Sensitivity analysis is a vital component of LP. By means of examples, we have illustrated the use of the Improved Algebraic Method (IAM) to efficiently derive the slack and surplus amounts for the resources of an LP. The parametric representation quickly provides all dual prices to carry out analysis for desirability of obtaining additional resources. The parametric representation also allows one to study versatility of the coefficients of the objective functions.

Managers face a variety of decision-making situations while balancing resources and meeting demand. The number of constraints on the recourses varies according to the product and the nature of problem. The IAM can be applied to a much larger set of linear inequalities than feasible by the standard algebraic approach and it enhances an understanding and presentation of simplex transparent.

Most companies do not maintain an objective to maximize or minimize within a planning time horizon because of too much pressure. Sometimes, the goal may be perceived as unattainable due to the changes required to achieve it. Managers often aim at an achievable goal. For example, a company might announce to their stockholders that they plan to reduce the operational cost by 10% in the third quarter. This is an incremental approach to optimization. In this paper, we demonstrated that the IAM can facilitate solving a goal-seeking problem efficiently.

In summary, we provide a simple algorithm which competes with the simplex and other LP software. The IAM contains the following nice features:

- It is an algebraic approach to solve a system of inequalities that provides a bridge between the graphical method and the simplex method, the LP software.
- It works within the decision variable space; no additional variables such as slack/surplus/artificial variables are added.
- It provides all information that the simplex method provides; such as shadow prices.
- It competes with LP software providing slack/surplus and sensitivity ranges for the RHS of the constraints and the coefficients of the objective function.
- It provides tight bounds on the objective function subject to given set of constraints.
- It finds a set of solutions to achieve a desirable value for the objective function.
- It is free from degeneracy which may cause cycling.
- It is easy to understand, easy to apply; therefore could prove valuable as a teaching tool.

Future work should extend the proposed method for application to an unbounded feasible region. Some areas for future research include looking for possible refinements, developments of an efficient code for performing a comparative computational study with other LP solvers. A suggested approach is to incorporate any symbolic software, such as Maple, into a computer algorithm to facilitate application to large problems.

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