

**TIGHT BOUNDING OF CONTINUOUS FUNCTIONS
OVER POLYHEDRONS: A NEW INTRODUCTION
TO EXACT GLOBAL OPTIMIZATION**

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Abstract: For some kinds of linearly constrained optimization problems with unique optimal solution, such as linear programs and convex problems, the single local optimum is also global. However, there are a broad variety of problems for which this property cannot be simply postulated or verified. The paper presents an effective approach for the global linearly constrained optimization problem with continuous objective function. With the help of a parametric representation of the feasible region an equivalent unconstrained problem is constructed which is much easier to solve. The classical optimization procedure is then applied to find the interior and boundary critical points. Evaluating the objective function at these critical points and the vertices identifies the global optimal solution. Our aim is to propose a new introduction to optimization, the design of a general solution algorithm that is easy for the user to understand and provides useful information such as global bounding of the objective function. The algorithm and its applications are presented in the context of some numerical examples solved by other methods.

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1. Introduction

Optimization has been of significant interest and relevance in many areas, also in engineering optimization [16, 20, 42, 44, 45, 50]. In particular, many design and operational problems give rise to linearly constrained optimization such as multi-objective linear programs [10, 49], entropy maximization, quadratic programs, and the dual of the geometric programs [14]. These typical problems are used extensively in the aerospace, automotive, electronic and chemical process industries. While the efficient computer packages of local solvers for these typical applications has become widespread, a major limitation is that there is often no guarantee that the generated solutions correspond to the global optima [30]. In many cases it means incurring a significant cost penalty, or even getting an incorrect solution to a physical design or operational process such as the calculation of chemical equilibria involving and ideal gas phase with many species and pure condensed phases.

Linearly constrained optimization problems are extremely varied. They differ in the form of the objective function, constraints, and in the number of variables. Although the structure of this problem is simple, finding a global solution – and even detecting a local solution is known to be difficult. The simplest form of this problem is realized when the objective function is linear. The resulting model is a linear program (LP). Other problems include fractional [3, 48], nonlinear network models [1, 4], quadratic [47], separable, geometric [14], convex [35] and nonconvex programs.

There are well over 400 different solution algorithms for solving different kinds of linearly constrained optimization problems. However, there is not one algorithm superior to others in all cases. For example in applying the Karush-Kuhn-Tucker (KKT) condition, it may be difficult, if not essentially impossible, to derive an optimal solution directly [32]. The most promising numerical solution algorithm is the feasible direction method, however, if objective function is nonconvex then the best one can hope for is that it converges to a local optimal point. Moreover, the optimum (local or global) may not be unique [9]. Therefore the question of finding global solutions to general optimization problems is an important one but as yet unanswered by general optimization theory in a practical way [41, p. 34].

Scope and Purpose. There are many engineering decisions that can be formulated as optimization problems and many algorithms to solve such problems. However, these algorithms are “custom-made” for each specific type of problem. Finding the global solution for general optimization problem is not an easy task. This paper proposes an effective explicit enumeration scheme for

solving a large class of problems with linear constraints and explicit continuous objective function. The key to the solution algorithm is removal of constraints through parametric representation of the problem. The global optimal solution is then found by computing the critical points and numerical evaluation of the objective function at these points as well as at the vertices.

As a by-product of the proposed solution algorithm, it enables us to compute the tight numerical bounds for a continuous objective function with the linear constraints. The existence of such tight bounds may depend on boundedness of feasible region and objective function.

Since the proposed solution algorithm is an enumeration method, it is the most effective (possibly not the most efficient) method for solving this type of problems, because, unlike other methods, it always finds the solution. The aim and scope of the paper is to introduce a new perspective for solving certain types of engineering optimization problems by a unified algorithm that always finds the solution. The contents are also useful to students and instructors teaching engineering optimization.

The remainder of this paper is organized as follows: Section 2 explains the algorithm and Section 3 an algebraic method for finding the vertices, edges and faces of the polyhedron, that can be used for the implementation of the method. The algorithm and its applications are presented in Section 4 in the context of small, hand-computation numerical problems each solved by other specialized algorithms in engineering optimization literatures. These numerical examples also provide some notions about the efficiency of the proposed solution algorithm. The proposed solution algorithm always finds the optimal solution successfully while in some presented cases other methods fail. The last section contains the conclusions with some useful remarks.

2. The Algorithm

We want to solve the following problem with linear feasible region:

Problem P. Maximize $f(\mathbf{x})$, Subject to: $\mathbf{Ax} \leq \mathbf{b}$, where some variables x_i have explicit upper and/or lower bounds and some are unrestricted in sign, where \mathbf{A} is $m \times n$ matrix, \mathbf{b} is m -vector and f is a continuous function. Problem **P** is a subset of a larger set of problems known as continuous global optimization problems, see [43].

The feasible region of the problem **P** is the set of points that defines the polyhedron [22, 55]. In the proposed solution we need to find all the critical points of objective function f inside and at the boundaries of the polyhedron.

A *polyhedron* with finite number of vertices can be represented in two equivalent ways [11]: H-representation and V-representation.

An *H-representation* of the polyhedron is given by an $m \times n$ matrix $\mathbf{A} = (a_{i,j})$ and m -vector $\mathbf{b} = (b_i)$:

$$S = \{\mathbf{x} \in R^n; \quad \mathbf{Ax} \leq \mathbf{b}\}.$$

An *V-representation* of the polyhedron is given by a minimal set of M vertices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M$ and N extreme rays $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_N$:

$$S = \left\{ \mathbf{x} \in R^n; \quad \mathbf{x} = \sum_{i=1}^M \lambda_i \mathbf{v}_i + \sum_{j=1}^N \mu_j \mathbf{w}_j, \quad \lambda_i, \mu_j \geq 0, \quad \sum_{i=1}^M \lambda_i = 1 \right\}.$$

A *face* of polyhedron S is a boundary set of S containing points on a line or plane (or hyper-plane). A *vertex* of this polyhedron is any of its points that can be specified as an intersection of faces. This is a point $\mathbf{v} \in R^n$ of S that satisfies an affinely independent set of n inequalities as equations. An *edge* of the polyhedron is the line segment between any two adjacent vertices. An *extreme ray* $\mathbf{w} \in R^n$ is a direction such that for some vertex \mathbf{v} and any positive scalar μ , $\mathbf{v} + \mu\mathbf{w}$ is in S and satisfies some set of $n - 1$ affinely independent inequalities as equations.

If a feasible region is bounded, then a corresponding polyhedron is called a *polytope* which has no extreme rays. Its V-representation is given by the convex combination of the vertices.

Example 1. The polyhedron in Figure 1, defined by

$$\begin{aligned} -x_1 + 2x_2 &\leq 2, \\ x_1 + x_2 &\leq 4 \end{aligned}$$

has one vertex and two extreme rays:

$$\mathbf{v}_1 = (2, 2), \quad \mathbf{w}_1 = (-2, -1), \quad \mathbf{w}_2 = (1, -1),$$

which indicates that its parametric representation is given by

$$(x_1, x_2) = (2 - 2\mu_1 + \mu_2, 2 - \mu_1 - \mu_2), \quad \mu_1, \mu_2 \geq 0.$$

For example, the unbounded edge defined by the vertex \mathbf{v}_1 and the extreme ray \mathbf{w}_1 can be represented as

$$(x_1, x_2) = (2 - 2\mu_1, 2 - \mu_1), \quad \mu_1 \geq 0.$$

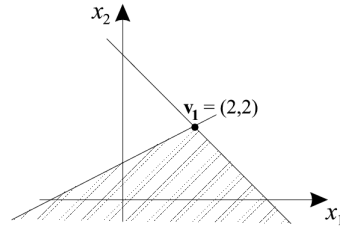


Figure 1: Unbounded polyhedron

The *parametric representation* of the objective function f is given by:

$$f(\mathbf{x}) = f(\mathbf{x}(\lambda, \mu)) = f(\lambda, \mu).$$

Critical point of a continuous function is a point where the first partial derivatives are zero or undefined.

In the proposed solution algorithm we need to find critical points. It is necessary for the domain to be an open set for the definition of derivative. Therefore, we solve unconstrained problems over some relevant open sub-domains of the feasible region. First, we find critical points on the interior points of the feasible region. Next, we evaluate the objective function at the vertices of the feasible region. Finally, we find critical points on interior of the faces and edges (i.e., line segments) of the feasible region. The global optimal solution is found by comparing the functional values at the critical points and at the vertices. Therefore, in solving an n dimensional problem, we solve some unconstrained optimization problems in $n, n - 1, \dots, 1$ dimensions. Thus, removing the constraints by the proposed algorithm reduces the constrained optimization to unconstrained problems which can be more easily dealt with.

The following provides an overview of the algorithm's process strategy:

Phase 1. Find the critical points of the objective function and select those which are feasible by checking the constraints.

Phase 2. Find the V-representation of the feasible region, its edges and faces. Evaluate the objective function at the vertices.

Phase 3. Find the critical points of the objective function over the open domains: faces, edges. Then evaluate the objective function at these points.

Phase 4. Pick the global solution and construct the numerically tight bounds for the problem.

The second phase of the algorithm can be implemented by one of the algorithms for finding the vertices, extreme rays, edges and faces of the polyhedron.

There are several approaches to the problem of generating all the vertices of the polyhedron. The double description method [38] involves building the polyhedron sequentially by adding the defining inequalities one at a time. Recent algorithms and practical implementations of this method have been developed by Fukuda and the others [19, 28, 29]. Another method for finding all the vertices and extreme rays of the polyhedron involves pivoting around the skeleton of the polyhedron. An efficient method using this approach is the reverse search method by Avis and Fukuda [13] and the revisited version [11]. Some other methods are described in [2, 12, 22, 25, 36, 54, 55]. The approach presented in [5] is based on an affine algebraic method, which is easy to understand and implement as described in the following section.

In Phase 3 of the algorithm we have to find the critical points over open domains. We can construct the parametric version of the objective function over each domain and look for its critical points. But since there may be many such domains it is more efficient to use the following procedure.

Suppose that the feasible region is defined by M vertices and N extreme rays. We will need partial derivatives of objective function f over each λ_i , $1 \leq i \leq M$ and each μ_j , $1 \leq j \leq N$. We can find them by using the chain-rule:

$$\begin{aligned}\frac{\partial f}{\partial \lambda_i} &= \sum_{k=1}^n \frac{\partial f}{\partial x_k} \cdot \frac{\partial x_k}{\partial \lambda_i}, \\ \frac{\partial f}{\partial \mu_j} &= \sum_{k=1}^n \frac{\partial f}{\partial x_k} \cdot \frac{\partial x_k}{\partial \mu_j}.\end{aligned}$$

Then, suppose the open domain is defined by a subset of the vertices $\mathbf{v}_1, \dots, \mathbf{v}_s$ and a subset of extreme rays $\mathbf{w}_1, \dots, \mathbf{w}_t$. To find the critical points on this domain we have to find the critical points of the parametric objective function over the domain

$$\lambda_1 + \dots + \lambda_s = 1, \quad \lambda_{s+1} = \dots = \lambda_M = \mu_{t+1} = \dots = \mu_N = 0,$$

$$0 < \lambda_1, \dots, \lambda_s < 1, \quad \mu_1, \dots, \mu_t > 0.$$

We can construct the Lagrangian

$$L(\lambda_1, \dots, \lambda_s, \mu_1, \dots, \mu_t, c) = f(\lambda_1, \dots, \lambda_s, \mu_1, \dots, \mu_t) + c(1 - \lambda_1 - \dots - \lambda_s)$$

In order to find the critical points we have to solve the following system:

$$\begin{aligned} \frac{\partial L}{\partial \lambda_1} &= \frac{\partial f}{\partial \lambda_1} - c = 0, \\ &\vdots \\ \frac{\partial L}{\partial \lambda_s} &= \frac{\partial f}{\partial \lambda_s} - c = 0, \\ \frac{\partial L}{\partial \mu_1} &= \frac{\partial f}{\partial \mu_1} = 0, \\ &\vdots \\ \frac{\partial L}{\partial \mu_t} &= \frac{\partial f}{\partial \mu_t} = 0, \\ \frac{\partial L}{\partial c} &= 1 - \lambda_1 - \dots - \lambda_s = 0. \end{aligned}$$

By eliminating c from the system we get

$$\begin{aligned} \frac{\partial f}{\partial \lambda_1} &= \frac{\partial f}{\partial \lambda_2} = \dots = \frac{\partial f}{\partial \lambda_s}, \\ \frac{\partial f}{\partial \mu_1} &= \frac{\partial f}{\partial \mu_2} = \dots = \frac{\partial f}{\partial \mu_t} = 0, \\ \lambda_1 + \lambda_2 + \dots + \lambda_s &= 1. \end{aligned}$$

It means that if we are looking for the critical points in the open domain defined by a subset of the vertices $\mathbf{v}_1, \dots, \mathbf{v}_s$ and a subset of extreme rays $\mathbf{w}_1, \dots, \mathbf{w}_t$, then we have to solve the system:

$$\begin{aligned} \frac{\partial f}{\partial \lambda_1} &= \frac{\partial f}{\partial \lambda_2} = \dots = \frac{\partial f}{\partial \lambda_s}, \\ \frac{\partial f}{\partial \mu_1} &= \frac{\partial f}{\partial \mu_2} = \dots = \frac{\partial f}{\partial \mu_t} = 0, \\ \lambda_1 + \lambda_2 + \dots + \lambda_s &= 1, \\ \lambda_i, \mu_j &\geq 0, \quad 1 \leq i \leq M, \quad 1 \leq j \leq N. \end{aligned}$$

In order to find the critical points in some open domain we can use the partial derivatives that were found once for all the domains.

In the last phase of the algorithm we have to compare the functional values at the critical points and the vertices. We pick the global solution. The proposed algorithm also enables us to compute the tight numerical bounds for a

continuous objective function with the linear constraints, because it discovers all critical points and vertices that are candidates for global minimums and maximums.

Not all problems have to go through all three steps. For example, if we are looking for a maximum, the objective function is concave and the interior critical point has been found, then we know that this is the optimal point.

3. Finding the Vertices and Domains of the Feasible Region

The graphical method of solving a system of linear inequalities, usually illustrated in textbooks, is limited to problems with one or two decision variables. However, it provides a clear visual understanding of the feasible region as well as the location of its vertices. While having a visual understanding of the problem is conducive, by an algebraic approach it becomes possible to solve higher dimension problems (i.e. three or more decision variables).

In this section we provide an easy to implement algebraic methodology for solving a linear system of inequalities to find all vertices of the feasible region. The methodology is applicable even when some or all of the decision variables are unrestricted in sign. The algebraic method for finding all the vertices of the feasible region is as follows:

Construct the Boundaries. Transform all inequalities (except sign-restrictions on decision variables, if any) to equalities by adding/subtracting slack/surplus variables. Construction of the boundary of the sign-restricted decision variables is included in the next step.

Find all Basic Solutions. Let T = the total number of variables including slack/surplus variables, E = the number of equations, and R = the total number of slack/surplus variables and sign-restricted decision variables. Set any $(T - E)$ variables to zero. The variables to be set to zero are the slack/surplus and sign-restricted decision variables (any $x_i \geq 0$, or $x_i \leq 0$) only. After setting $(T - E)$ variables to zero, solve the resulting squared system of equations to obtain the values of the remaining variables. Note that the maximum number of basic solutions is:

$$\frac{R!}{(T - E)!(R + E - T)!},$$

where the symbol ! stand for Factorial.

Check For Feasibility. All slack/surplus variables must be nonnegative. The sign-restriction on each decision variable, if any, must be satisfied. The

obtained set of basic feasible solutions constitutes the vertices of the bounded feasible region defined by the system.

Identification of Domains of the Feasible Region. As pointed out earlier, if the objective function is nonlinear, the optimal solution to problem **P** may be one of the stationary points of the feasible region. We need to identify all relevant open domains of the feasible region to compute all stationary points using the gradient. The interior of the feasible region can be defined by the full set of the vertices obtained. The interior of other relevant domains such as the faces, edges, etc. of the feasible region can be defined by appropriate subsets of these vertices. Accordingly, we present the following method to identify all such subsets of vertices through a constraint-vertex table.

Let n be the number of decision variables in the problem formulation. Construct a table with one column for each vertex and one row for each constraint including the sign-restriction conditions, if any, on the decision variables. Record in each cell of the table whether that vertex binds that constraint or not. First, obtain the sets of vertices that bind any one common constraint; each set so obtained defines a face in a three dimensional case and an edge in a two dimensional case. Next, obtain the sets of vertices that bind any two common constraints; each set so obtained defines an edge in the three dimensional case. Third, obtain the sets of vertices that bind any three common constraints, and so on; but, not beyond $(n - 1)$ common constraints.

Example 2. Consider the following feasible region:

$$\begin{aligned}x_1 + x_2 + x_3 &\leq 10, \\3x_1 + x_3 &\leq 24, \\x_1, x_2, x_3 &\geq 0.\end{aligned}$$

The vertices, edges and faces of this feasible region are graphically illustrated in Figure 2.

Note that, in Figure 2, the feasible region is defined by a set of five constraints including the sign-restrictions on the three decision variables. The feasible region has six vertices which can be found by the algebraic method presented earlier as is shown in Table 1.

Therefore, the vertices of the feasible region are:

$$\begin{aligned}\mathbf{v}_1 &= (8, 0, 0), \quad \mathbf{v}_2 = (8, 2, 0), \quad \mathbf{v}_3 = (0, 10, 0), \\ \mathbf{v}_4 &= (0, 0, 10), \quad \mathbf{v}_5 = (7, 0, 3), \quad \mathbf{v}_6 = (0, 0, 0).\end{aligned}$$

Construction of the constraint-vertex table: Note that, in Figure 2, the feasible region has five faces, nine edges and, of course, one interior. Each vertex

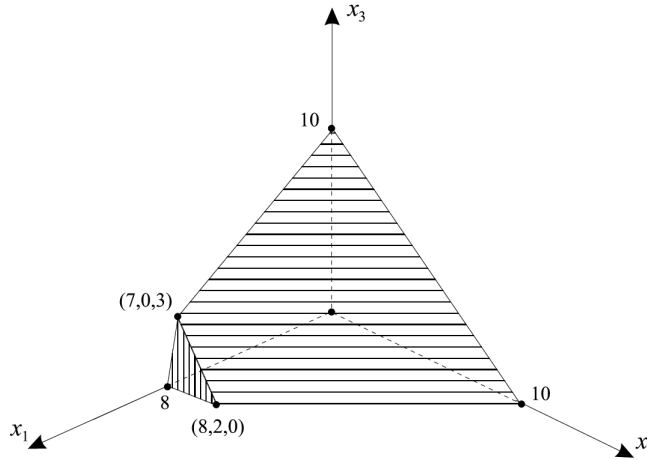


Figure 2: Feasible region for Example 2

is uniquely defined by three constraints at binding position. All vertices binding one common constraint define each face. All vertices binding two common constraints define each edge. And, of course, all six vertices together define the interior of the feasible region. Table 2 shows the construction of constraint-vertex table.

Vertices binding one common constraint (i.e. faces of the feasible region) are shown in Table 3 and vertices binding two common constraints (i.e., edges of the feasible region) in Table 4.

4. Numerical Examples

In the following examples we demonstrate how the proposed algorithm can be used to solve general linearly constrained optimization problems.

Example 3. The following quadratic optimization with inequality constraint is attempted to solve in [23, pp. 70-82] using the Wolf Method.

$$\text{Max } f(x_1, x_2) = 2x_1 + x_2 + 3x_1x_2 - x_1^2 - 2x_2^2,$$

$$\begin{aligned} \text{subject to: } x_1 + 2x_2 &\leq 10, \\ x_1 + 3x_2 &\geq 3, \\ x_1, x_2 &\geq 0. \end{aligned}$$

| x_1 | x_2 | x_3 | s_1 | s_2 | Feasible? |
|-------|-------|-------|-------|-------|------------|
| 8 | 0 | 0 | 2 | 0 | yes |
| 8 | 2 | 0 | 0 | 0 | yes |
| 0 | 10 | 0 | 0 | 24 | yes |
| 0 | 0 | 10 | 0 | 14 | yes |
| 0 | 0 | 24 | -14 | 0 | no |
| 10 | 0 | 0 | 0 | -6 | no |
| 7 | 0 | 3 | 0 | 0 | yes |
| 0 | -14 | 24 | 0 | 0 | no |
| 0 | 0 | 0 | 10 | 24 | yes |

Table 1: Finding the vertices by the algebraic method

| Constraint | v_1 | v_2 | v_3 | v_4 | v_5 | v_6 |
|------------------------|-------|-------|-------|-------|-------|-------|
| $x_1 + x_2 + x_3 = 10$ | no | yes | yes | yes | yes | no |
| $3x_1 + x_3 = 24$ | yes | yes | no | no | yes | no |
| $x_1 = 0$ | no | no | yes | yes | no | yes |
| $x_2 = 0$ | yes | no | no | yes | yes | yes |
| $x_3 = 0$ | yes | yes | yes | no | no | yes |

Table 2: Construction of constraint-vertex table

The partial derivatives of the objective function are

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= 2 - 2x_1 + 3x_2, \\ \frac{\partial f}{\partial x_2} &= 1 + 3x_1 - 4x_2.\end{aligned}$$

The gradient vanishes at $x_1 = -11$, $x_2 = -8$ which is not feasible. It means that there are no feasible interior critical points.

The vertices of the feasible region and the corresponding objective function values are listed in Table 5.

The edges of the polyhedron are

$$\mathbf{e}_1 = (\mathbf{v}_1, \mathbf{v}_2), \mathbf{e}_2 = (\mathbf{v}_2, \mathbf{v}_3), \mathbf{e}_3 = (\mathbf{v}_3, \mathbf{v}_4), \mathbf{e}_4 = (\mathbf{v}_1, \mathbf{v}_4).$$

The parametric representation of the feasible region is:

$$(x_1, x_2) = (3\lambda_1 + 10\lambda_2, 5\lambda_3 + \lambda_4),$$

| One common constraint | Vertices |
|------------------------|----------------------|
| $x_1 + x_2 + x_3 = 10$ | v_2, v_3, v_4, v_5 |
| $3x_1 + x_3 = 24$ | v_1, v_2, v_5 |
| $x_1 = 0$ | v_3, v_4, v_6 |
| $x_2 = 0$ | v_1, v_4, v_5, v_6 |
| $x_3 = 0$ | v_1, v_2, v_3, v_6 |

Table 3: Faces of the feasible region

| Two common constraints | Vertices |
|---|------------|
| $x_1 + x_2 + x_3 = 10, 3x_1 + x_3 = 24$ | v_2, v_5 |
| $x_1 + x_2 + x_3 = 10, x_1 = 0$ | v_3, v_4 |
| $x_1 + x_2 + x_3 = 10, x_2 = 0$ | v_4, v_5 |
| $x_1 + x_2 + x_3 = 10, x_3 = 0$ | v_2, v_3 |
| $3x_1 + x_3 = 24, x_1 = 0$ | No edge |
| $3x_1 + x_3 = 24, x_2 = 0$ | v_1, v_5 |
| $3x_1 + x_3 = 24, x_3 = 0$ | v_1, v_2 |
| $x_1 = 0, x_2 = 0$ | v_4, v_6 |
| $x_1 = 0, x_3 = 0$ | v_3, v_6 |
| $x_2 = 0, x_3 = 0$ | v_1, v_6 |

Table 4: Edges of the feasible region

where $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$, $0 \leq \lambda_1, \lambda_2, \lambda_3, \lambda_4 \leq 1$.

For finding the critical points on the edges we will need partial derivatives of f over $\lambda_1, \lambda_2, \lambda_3$ and λ_4 . By using the chain-rule we get:

$$\begin{aligned} \frac{\partial f}{\partial \lambda_1} &= 3 \frac{\partial f}{\partial x_1} = 6 - 6x_1 + 9x_2, \\ \frac{\partial f}{\partial \lambda_2} &= 10 \frac{\partial f}{\partial x_1} = 20 - 20x_1 + 30x_2, \\ \frac{\partial f}{\partial \lambda_3} &= 5 \frac{\partial f}{\partial x_2} = 5 + 15x_1 - 20x_2, \\ \frac{\partial f}{\partial \lambda_4} &= \frac{\partial f}{\partial x_2} = 1 + 3x_1 - 4x_2. \end{aligned}$$

To find the critical points on the interior of the edge $\mathbf{e}_2 = (\mathbf{v}_2, \mathbf{v}_3)$ we have

| vertex | its coordinates | $f(\mathbf{x})$ |
|----------------|-----------------|-----------------|
| \mathbf{v}_1 | (3,0) | -3 |
| \mathbf{v}_2 | (10,0) | -80 |
| \mathbf{v}_3 | (0,5) | -45 |
| \mathbf{v}_4 | (0,1) | -1 |

Table 5: Vertices and function values at the vertices for Example 3

to solve the system

$$\frac{\partial f}{\partial \lambda_2} = \frac{\partial f}{\partial \lambda_3},$$

over the domain $\lambda_2 + \lambda_3 = 1, 0 < \lambda_2, \lambda_3 < 1, \lambda_1 = \lambda_4 = 0$. We get

$$20 - 20x_1 + 30x_2 = 5 + 15x_1 - 20x_2.$$

Since $x_1 = 10\lambda_2$ and $x_2 = -5\lambda_2 + 5$, the critical point is $x_1 = 53/12, x_2 = 67/24$ with the objective function value 13.52.

There is one more critical point on the edge $\mathbf{e}_4 = (\mathbf{v}_1, \mathbf{v}_4)$ with objective function value 3.05. On the remaining edges there are no critical points.

Therefore, the optimal solution occurs at $x_1 = 53/12$, and $x_2 = 67/24$ with an optimal value of 13.52. The solution given in the above reference is $x_1 = 9/8$, and $x_2 = 5/8$ with objective value of $47/16 = 2.94$ which is inferior compared with the global solution obtained by the proposed approach.

Tight bounds of the objective function over the feasible region are:

$$-80 \leq f(x_1, x_2) \leq 13.52.$$

Example 4. The following non-linear fractional program is from [39], which is solved by a specialized algorithm therein.

$$\text{Min } f(x_1, x_2, x_3, x_4) = \frac{(2 - x_1 - x_2 - x_3 - x_4)^2}{(2 - x_1)^2},$$

$$\begin{aligned} \text{subject to: } x_1 + x_2 + x_3 + x_4 &= 1.5, \\ x_1 - x_3 &= 0, \\ x_1 + x_2 - x_4 &= 0, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

From a study of the feasible region, we realize that the denominator of f does not vanish, therefore the problem is a continuous optimization. The

feasible region has two vertices (Table 6), found by solving 3 equations with 4 unknowns while setting at least any one variable to zero then checking for feasibility by substitution.

| vertex | its coordinates | $f(\mathbf{x})$ |
|----------------|----------------------|-----------------|
| \mathbf{v}_1 | $(0, 1/2, 1/2, 1/2)$ | $1/16$ |
| \mathbf{v}_2 | $(3/4, 0, 0, 3/4)$ | $4/25$ |

Table 6: Vertices and function values at the vertices for Example 3

The feasible region consists of a line segment joining the two distinct vertices. Therefore, the parametric representation of the feasible region is:

$$(x_1, x_2, x_3, x_4) = (3/4\lambda_2, 1/2\lambda_1, 1/2\lambda_1, 1/2\lambda_1 + 3/4\lambda_2),$$

for all $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$.

By substituting $\lambda_1 = 1 - \lambda_2$, the parametric representation of the objective function for the interior points is:

$$f(\lambda_2) = \frac{1}{(8 - 3\lambda_2)^2},$$

over the open domain $0 < \lambda_2 < 1$. The derivative of $f(\lambda_2)$ does not vanish within its domain, therefore there is no interior critical points for this problem.

Evaluating the objective function at vertices indicates that the optimal solution is at the vertex \mathbf{v}_1 with objective function value of $1/16$. Therefore, the objective function is bounded:

$$1/16 \leq f(x_1, x_2, x_3, x_4) \leq 4/25,$$

over the feasible region.

Example 5. This example shows the use of proposed algorithm in the case of unbounded feasible region. The problem is from [42, pp. 257-263], which is solved therein by the generalized reduced gradient method. We are looking for

$$\min f(x_1, x_2) = -2x_1 - 4x_2 + x_1^2 + x_2^2 + 5$$

subject to the same constraints as in Example 1. The feasible region for this example is drawn on Figure 1.

The gradient of the objective function is $(-2 + 2x_1, -4 + 2x_2)$. It vanishes at $x_1 = 1$, $x_2 = 2$ which is not feasible. Therefore there is no interior critical point for this problem.

Feasible region has only one vertex $\mathbf{v}_1 = (2, 2)$ with the objective function value of 1. The polyhedron is unbounded and has two unbounded edges:

$$\mathbf{e}_1 = (\mathbf{v}_1, \mathbf{w}_1), \mathbf{e}_2 = (\mathbf{v}_2, \mathbf{w}_2).$$

Parametric representation of the feasible region is given by

$$(x_1, x_2) = (2 - 2\mu_1 + \mu_2, 2 - \mu_1 - \mu_2), \quad \mu_1, \mu_2 \geq 0.$$

Partial derivatives of the objective function over each μ are

$$\begin{aligned} \frac{\partial f}{\partial \mu_1} &= 8 - 4x_1 - 2x_2, \\ \frac{\partial f}{\partial \mu_2} &= 2 + 2x_1 - 2x_2. \end{aligned}$$

To find the critical points on the interior points of unbounded edge $\mathbf{e}_1 = (\mathbf{v}_1, \mathbf{w}_1)$, where $\mathbf{w}_1 = (-2, -1)$ is an extreme ray, we have to find the solution of

$$\frac{\partial f}{\partial \mu_1} = 8 - 4x_1 - 2x_2 = 0,$$

over the domain $\mu_1 > 0, \mu_2 = 0$.

Since $x_1 = 2 - 2\mu_1$ and $x_2 = 2 - \mu_1$, we find the critical point $x_1 = 6/5$, $x_2 = 8/5$ with the objective function value of $1/5$. Similarly, we find out that there is no critical point on the remaining edge.

Because we are looking for a minimum, the global optimal solution is: $x_1 = 1.2$ and $x_2 = 1.6$ with the optimal value 0.2. The problem has no upper bound.

More applications of the proposed solution algorithms to engineering optimization are available in [8], including problems from [46], pp. 837-858; [21]; [44], pp. 463-465; and [50], pp. 86-92. In almost all cases, the proposed method produces better solution than found using the specialized and diverse techniques therein.

5. Conclusion

We have presented a new solution algorithm for the linearly constrained global optimization problems with continuous objective function. For a polyhedron specified by a set of linear equalities and/or inequalities, the proposed solution algorithm utilizes its parametric representation. This parametric representation of the feasible region enables us to solve a large class of optimization problems.

The key to this generalized solution algorithm is that the constrained optimization problem is converted to an unconstrained optimization problem through a parametric representation of the feasible region.

It favorably compares with other methods for this type of problems. The proposed algorithm, unlike other general purpose solution methods, such as generalized reduced gradient, guarantees globally optimal solutions, it has simplicity, potential for wide adaptation, and deals with all cases. However, this does not imply that all distinction among problems should be ignored. One can incorporate the special characteristic of the problem to modify the proposed algorithm in solving them.

While the Lagrange and KKT (penalty-based) methods “appear” to remove the constraints by using a linear (or nonlinear) combination of the constraints in a penalty function, the proposed solution algorithm, however, uses the linear convex combination of vertices to remove the constraints. The main drawback for the proposed algorithm is that all the vertices of the feasible region have to be found.

The main advantages of the presented algorithm are that it covers all linearly constrained optimization problems and it always finds the optimal solution. There are many problems in the literature for which the proposed algorithm finds optimal solution and others do not.

A computerized version with some possible refinements is needed to handle large-scale problems and special cases, such as incorporating efficient and effective numerical techniques for finding the critical points for implicit functional problems. An immediate future work is to study other algorithms and to test and develop efficient computerized version of our proposed effective method.

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