Global Optima for Linearly Constrained Business Decision Models

Hossein Arsham
Division of Economics, Finance, and Management Science
University of Baltimore,USA

Abstract :

It is well-known that many business administration decision problems can be formulated as optimization problems. There are well over four hundred algorithms to solve such problems. However, these algorithms are custom-made for each specific type of problem. This has lead to classification of problems, such as linear, fractional, quadratic, convex and non-convex programs. This paper presents a simple alternative approach to obtain global solutions to bounded linearly constrained optimization problems with differentiable objective functions. We propose an effective explicit enumeration scheme for solving a large class of problems with linear constraints and a differentiable objective function. The primary intention of this paper is to provide an optimization tool that can be understood easily and applied to a wide range of problems. The unified approach is accomplished by converting the constrained optimization problem to an unconstrained optimization problem through a parametric representation of its feasible region. The proposed algorithm has the following useful features. It is a general-purpose algorithm; i.e., it employs one common treatment for all cases; it guarantees global optimization in each case unlike other general-purpose local optimization algorithms; it has simplicity because it is intuitive and requires only first order derivatives (gradient); and it provides useful information for sensitivity analysis. The solution algorithm and its applications to finance, economics, marketing, and production and operations management are presented in the context of numerical problems already solved by other methods.

Keywords: Business optimization, Financial portfolio selection, Linearly constrained global optimization, mathematical methods and programming.

JEL Classification: A2; C6; I22.

The Author is most appreciative to Professors O. Popjoy and M. Sultani for their useful suggestions and comments. This work is supported by the National Science Foundation under grant CCR-9505732.

1. Introduction

Many business administration problems lend themselves to the application of constrained optimization techniques. From the manager's perspective, a better understanding of optimization techniques would prove most helpful in arriving at well-reasoned decisions. Unfortunately, even though most managers and business students have the basic calculus skills necessary to understand and learn such techniques, training in optimization theory is receiving little attention.

Problem Statement:

Global optimization is concerned with the characterization and computation of global minima or maxima of nonlinear functions. Such problems are widespread in mathematical modeling of real world systems for a very broad range of applications, Thi et al (2002).

We define a standard global optimization problem as follows, Horst *et al* (1995):

Given a nonempty set $D \subset \Re^n$ *and a continuous function* f *find at least one point* $x^* \in D$ *satisfying* $f(x^*) \leq f(x)$ *for all* $x \in D$*, or show that such a point does not exist.*

Standard optimization techniques have not been successful in solving these problems because they use only local information and hence cannot be expected to provide global optimality criteria Horst and Tuy (1996) and Barrientos et al (2000). Such algorithms usually obtain a local minimum that is global only when certain conditions are satisfied, such as objective function and feasible region being convex. Moreover, the problem of checking local optimality for a feasible point and the problem of checking if a local minimum is strict are not easy tasks. Other problems regarding global optimization are that the optimum is often attained at the boundary of the feasible region and the optimum need not be unique.

In this paper we wish to present an alternative approach for solving linearly constrained global optimization problems that are the subset of general global optimization problems:

Problem P: Max f(**X**)
subject to: $\mathbf{AX} \leq \mathbf{a}$, $\mathbf{BX} \geq \mathbf{b}$, $\mathbf{DX} = \mathbf{d}$,
with the possibility that some decision variables $X_j \geq 0$, some $X_j \leq 0$, and some X_j unrestricted in sign

where f(X) is a continuous function, and the matrices A, B, and D and the vectors a, b, and d have appropriate dimensions.

Indeed, linearly constrained optimization problems are extremely varied. They differ in the form of their objective function, the constraints, and in the number of decision variables. Moreover, the general optimization represented by Problem P has very diverse applications. Linearly constrained problems are typical of the business optimization models. These business decision models include the transportation problems, Arsham (1992), project management, Arsham (1993), network optimization, Arsham (1995), fractional optimization, Arsham (1990), product variety and ordering, Jayaraman and Srivastava (1998), and grouping of customers for better allocation of resources, Tyagi and Das (1999).

Although the structure of Problem P is simple, finding the global optimal solution - and even detecting a local optimal solution - is well-known to be difficult. The simplest form of this problem is realized when the function f(**X**) is linear. The resulting model is a linear program (LP). Other problems, where f(**X**) is nonlinear, include fractional, quadratic, convex and non-convex programs. There are well over four hundred different algorithms for solving different kinds of linearly constrained optimization problems. However, there does not exist an algorithm that is generally superior to all others, nor does any one algorithm guarantee global optimal solution in all cases. Another problem regarding global optimization is that optimum is often attained at the boundary of feasible region and that more than one optimum can exist.

In some cases, applying the optimality conditions, such as the

KarushKuhnTucker conditions, see e.g., Hillier and Lieberman (1995), it may be difficult, if not impossible, to derive the optimal solution. The most promising solution algorithm appears to be the feasible direction method. However, if f(**X**) is non-convex, then the best one can hope for is that it converges to a local optimal solution.

In the proposed solution approach, if the objective function f(**X**) is nonlinear, we need to find all stationary points. However, since the gradient (first order derivative) of a function is not defined at the boundary on a closed domain, we solve unconstrained problems over some relevant open domains. These relevant open domains are identified through a constraint-vertex table. First, we find stationary points in the *interior* of the bounded feasible region. Next, we find stationary points in the interior of the *faces* of the feasible region, then in the interior of the *edges* of the feasible region. Finally, by evaluation of the objective function at the stationary points and at the vertices of the feasible region, the global optimal solution is found. Thus, the removal of the constraints by the proposed algorithm reduces the constrained optimization problem to some unconstrained optimization problems, which can be easily dealt with using the gradient.

An algorithm that is generally superior to all others does not exist, nor does any one algorithm guarantee global optimal solution in all cases. It requires continuous objective functions, however, it allows the possibility that the objective function may be non-differentiable at a finite number of feasible points, i.e., almost differentiable objective function.

Since problem P is too complex and cannot be solved generally by the existing methods, we propose a unified general-purpose solution algorithm to find the global solution.

The remainder of the paper is organized as follows. In the following section we present the current state of optimization applications in business administration. Section 3 presents the algebraic method and the constraint-vertex table. The parametric representation and the solution strategy are developed in Section 4. The applications demonstrating its ease of use to finance, economics, marketing, and production are presented in Section 5. The last section provides concluding comments and some useful remarks.

2. Current State of Optimization Applications in Business Administration

This section presents the current state of optimization applications in finance. However its findings apply also to economics, marketing, and production and operations management areas, see e.g., Arsham and Stanton (2003).

Almost all finance textbooks avoids the complexities of the allocation problem entirely, and use pictorial approaches to explain how the frontier of efficient investment portfolios can be constructed. By plotting the expected returns of all possible portfolios in relation to their respective variances, and by drawing an envelope around the portfolios that has the largest return for any level of variance. Doing so, a graph emerges that depicts the portfolios that generate the greatest expected return for any given level of variance. Although this approach describes the nature of the problem, the investor's choice of portfolio weights remains a second-stage decision. What is not pointed out is that the investor would need to compute the returns and variances of an infinite number of portfolios to implement this approach. Even if the decision-maker is able to construct the frontier using this pictorial approach, the actual selection process requires the additional step of constructing a map of the investor's indifference curves and finding a tangency between the indifference curves and the efficient set of portfolios. Furthermore, the challenges presented in constructing the indifference map are non-trivial.

A critical assessment of the more advanced textbooks reveals that most of the mathematical details provided are relegated to appendices. For the most part, the explanations offered are prohibitively complex, difficult to understand, computationally tedious, generally just descriptive in nature, and often provide more questions than answers. More troubling is that even with a higher level of mathematical content, key components of the approaches may be omitted on the grounds that they are beyond the scope of the textbook. As an example, Elton et al (2003) outline the critical-line approach to solving the investment problem. This algorithm requires the determination of corner points at which the objective function changes direction, but there is no explanation as to how these corner points are determined. The authors provide interested readers with a reference that explains how to find the corner points, but without the corner points the algorithm can not be implemented. In a similar fashion, Sharpe et al (1999) claim that the critical-line method is beyond the scope of the book.

It may be feasible for students to learn how to apply Karush-Kuhn-Tucker (KKT) conditions to solve problems of this type, but the necessary knowledge to do so correctly –in particular, to recognize when the conditions are violated– may be challenging. Elton et al. (2003) provide examples that apply KKT conditions, but, without a substantial amount of effort, the students' level of understanding is unlikely to be enhanced, and in any case, the approach is quite laborious.

Taking all of the existing textbooks together, they still fail to provide easy to implement decision-making tools and do little to advance the intuition underlying constrained optimization problems. This is most unsatisfactory indeed. The end result is that the current textbook treatment does not provide students with an approachable means of mastering this important and useful topic. This trend is unfortunate because decision-making skills and intuition can be greatly improved at relatively low intellectual cost.

We consider the manager's primary objective is a methodology that provides a prescriptive, rather than a purely descriptive, solution. To make this workable, it is crucial to provide an explanation that is simpler to teach and understand, that will enhance the reader's ability to make better decisions, and that will provide the intuition necessary to understand complex optimization problems. The key to addressing this problem is to provide a simple and easily implemented algorithm that requires little mathematical sophistication. Once a decision-maker has mastered this approach, he or she will have a far better understanding of optimization problems in general, and this will allow them to evaluate more adequately any proposed solutions presented by better-trained analysts or other consultants. It is possible to provide a bridge between an analyst with a high level of mathematical training and a manager faced with the difficulty of making the final decision.

This paper presents a simple algorithm for determining the globally optimal solution to linearly constrained optimization problems. It requires only the most basic mathematical skills; so, it is approachable by a wide range of business students. Obvious applications include portfolio selection, economics, marketing, and production problems. The algorithm will prove useful also as an introduction to optimization prior to introducing more advanced topics.

The proposed algorithm relies on evaluation of the objective function. The need to compute the Hessian is eliminated. Consequently, managers and students who are not mathematics majors easily understand it. The only prerequisite is an understanding of the basic calculus needed to find critical points of a function. In addition to its overall simplicity, the algorithm has several other attractive features. It is a general-purpose approach applicable to all linearly constrained optimization problems. And, unlike other methods, such as the KKT conditions, it guarantees that the solution is globally optimal. The algorithm also provides useful information such as all of the critical points of the objective function which in turn, provide tight upper and lower bounds over the feasible region.

3. The Identification of the Vertices, Edges, and Faces of the Bounded Feasible Region

As pointed out earlier, if the objective function is nonlinear, the optimal solution to Problem P may be one of the stationary points on the interior, faces, edges, etc. of the feasible region in addition to the vertices. The interior of the feasible region is defined by the full set of the vertices obtained. Other relevant domains, such as faces, edges, etc. of the feasible region are defined by appropriate subsets of these vertices. Accordingly, we present the following method to identify all such subsets of vertices using a constraintvertex table. consider the following feasible region:

$$X_1 + X_2 + X_3 \leq 10,\ 3X_1 + X_3 \leq 24,\ X_1 \geq 0,\ X_2 \geq 0,\ X_3 \geq 0.$$

The vertices, edges and faces of this feasible region are shown in the following figure.

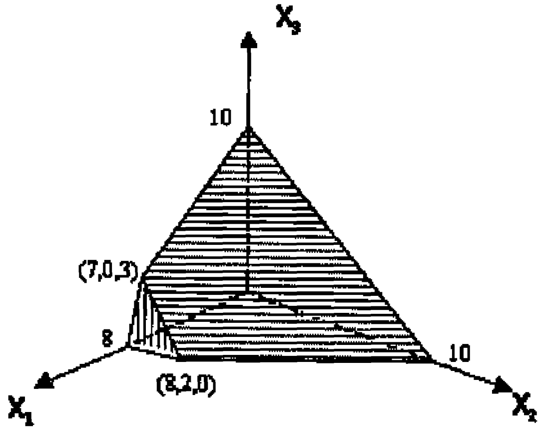

Figure 1: The Feasible Region for the Numerical Example

Note in Figure 1 that the feasible region is defined by a set of five constraints, which includes the three signed-variables. The feasible region has six feasible vertices. The coordinates of these vertices are the basic feasible solution of the systems of equations obtained by setting some of the constraints at binding (i.e., equality) position. Therefore, by taking any three of the equations and solving them simultaneously one obtains a basic solution (if exists). By plugging this basic solution in the constraints of other equations, one can check for feasibility of the basic solution. If it is feasible, then this solution is a basic feasible solution that provides the coordinates of a corner point of the feasible region. For a bounded feasible region, the number of vertices is at most combinatorial C_p^q where p is the number of constraints and q is the number of variables, $p \geq q$. Since in the large-scale problems the number of vertices could be huge, the proposed algorithm in this paper is well suited for small-size problem such as those appeared often in business textbooks.

To illustrate the procedure, consider all of the constraints at binding position, i.e., all with equality (=) sign. This produces the following equations:

$X_1 + X_2 + X_3 = 10$
$3X_1 + X_3 = 24$
$X_1 = 0$
$X_2 = 0$
$X_3 = 0.$

Here we have p=5 equations with q=3 unknowns. In terms of a "binomial coefficient", there are at most $C_5^3 = 5! / [3! (5-3)!] = 10$ basic solutions. Solving the six resultant systems of equations, we have:

Table 3.1: All the Basic Solutions

Therefore, there are m = 6 vertices, denoted by (Vi, i = 1, .., 6) are:

Equations	(X1, X2, X3)	Constraints Checking	Feasible?
1, 2, 3	0, -14, 24	4, 5	No
1, 2, 4	7, 0, 3	3, 5	Yes
1, 2, 5	8, 2, 0	3, 4	Yes
1, 3, 4	0, 0, 10	2, 5	Yes
1, 3, 5	0, 10, 0	2, 4	No
1, 4, 5	10, 0, 0	2, 3	Yes
2, 3, 4	0, 0, 24	1, 5	No
2, 3, 5	No solution	----	No
2, 4, 5	8, 0, 0	1, 3	Yes
3, 4, 5	0, 0, 0	1, 2	Yes

Table 3.2: All the Feasible Vertices

V1	$X_1 = 7$	$X_2 = 0$	$X_3 = 3$
V2	$X_1 = 8$	$X_2 = 2$	$X_3 = 0$
V3	$X_1 = 0$	$X_2 = 0$	$X_3=10$
V4	$X_1=10$	$X_2 = 0$	$X_3= 0$
V5	$X_1 = 8$	$X_2 = 0$	$X_3 = 0$
V6	$X_1 = 0$	$X_2 = 0$	$X_3 = 0$

Construction of the ConstraintVertex Table:

Let n be the number of decision variables in the problem formulation. Construct a table with one column for each vertex and one row for each constraint, including the signrestriction conditions on the decision variables. Record in each cell of the table whether that vertex binds that constraint or not. First, obtain the sets of vertices that bind any one common constraint; each set so obtained defines a face in a three dimensional case and an edge in a two dimensional case. Next, obtain the sets of vertices that bind any two common constraints; each set so obtained defines an edge in the three dimensional case. Third, obtain sets of vertices that bind any three common constraints, and so on, but, not beyond (n1) common constraints.

For our numerical example note that, the feasible region is defined by a set of five constraints, which includes the three restricted-in-sign decision variables. The feasible region has six vertices that represent the basic feasible solutions. The feasible region has five faces, nine edges, and, of course, one interior set. Each vertex is uniquely defined by three constraints at binding (i.e., equality) position. Vertices binding one common constraint define each face. Vertices binding two common constraints define each edge. And, of course, all six vertices together define the interior of the feasible region.

Constraints	Vertex Number V(i) 1	2	3	4	5	6
$X_1 + X_2 + X_3 \leq 10$	yes	yes	yes	yes	no	no
$3X_1 + X_3 \leq 24$	yes	yes	no	no	yes	no
$X_1 \geq 0$	no	no	yes	yes	no	yes
$X_2 \geq 0$	yes	no	yes	no	yes	yes
$X_3 \geq 0$	no	yes	no	yes	yes	yes

Table 3.3: The ConstraintsVertex Table

All the vertices that bind one common constraint provide a face of the feasible region:

One Common Constraint	Vertices V(i)
$X_1 + X_2 + X_3 = 10$	1, 2, 3, 4
$3X_1 + X_3 = 24$	1, 2, 5
$X_1 = 0$	3, 4, 6
$X_2 = 0$	1, 3, 5, 6
$X_3 = 0$	2, 4, 5, 6

Table 3.4: Identification of Faces

Vertices that bind two common constraints provide an edge of the feasible region:

Two Common Constraints	Vertices V(i)
$X_1 + X_2 + X_3 \leq 10$, $3X_1 + X_3 \leq 24$	1, 2
$X_1 + X_2 + X_3 \leq 10$, $X_1 \geq 0$	3, 4
$X_1 + X_2 + X_3 \leq 10$, $X_2 \geq 0$	1, 3

$X_1 + X_2 + X_3 \le 10, X_3 \ge 0$	2, 3
$3X_1 + X_3 \le 24, X_1 \ge 0$	no edge
$3X_1 + X_3 \le 24, X_2 \ge 0$	1, 5
$3X_1 + X_3 \le 24, X_3 \ge 0$	2, 5
$X_1 \ge 0, X_2 \ge 0$	3, 6
$X_1 \ge 0, X_3 \ge 0$	4, 6
$X_2 \ge 0, X_3 \ge 0$	5, 6

Table 3.4: Identification of Edges

4. Parametric Representation and the Solution Strategy

Unlike linear programs, the optimal solution to a nonlinear problem P need not always be at a vertex of the feasible region. Let us associate a parametric variable λ_i with each vertex V_i of the bounded feasible region for all i = 1, 2,....., m, where m is the number of vertices of the feasible region. The parametric representation of the decision variables:

$$\mathbf{X} = \sum_{i=1}^{m} \lambda_i V_i,$$

where

$$\sum_{i=1}^{m} \lambda_i = 1,$$

$$\lambda_i \ge 0,$$

$$\forall i = 1,2,\ldots,m$$

defines any point of the bounded feasible region. Substituting **X** = in the objective function, f(**X**), the parametric representation of problem P is:

Problem P: Max f(λ)

subject to: $\sum_{i=1}^{m} \lambda_i = 1$

$\lambda_i \ge 0,$

$\forall$ i =1, 2, ..., m.

In the proposed solution algorithm, we need to find critical points. Notice that we distinguish between stationary points and critical points. A *critical point* of a continuous function is any point where the first partial derivatives are zero or undefined, while *stationary point* is generally used to mean a KKT point. In an unconstrained problem, a stationary point refers to the point where the gradient of the objective function is zero. Therefore, the set of stationary points is a subset of the critical points set for a continuous function.

The following provides an overview of the solution strategy:

1. *Find the interior critical points by using the gradient of the objective function, and then select those that are feasible by checking the constraints.*

2. *Find vertices of the feasible region. Arsham (1997) provides a simple and direct algebraic methodology for solving a linear system of inequalities by finding vertices of the feasible region without using slack/surplus variables.*

3. *Compute all other critical points first by constructing the para - metric representation of the objective function over the open domains of the boundaries of the feasible region, and then by using its gradient, or by using directly the chainrule for the construction of the paramer - ic gradient.*

4. *Evaluate the objective function at critical points and vertices.*

5. *Select the global solution.*

Since we employed a convex parametric transformation, therefore, if Problem P has a global optimal solution, then f(λ) has a global optimal solution. The following steps summarize the implementation of the proposed solution algorithm:

Step 1. Evaluate f(λ) at its stationary points within the interior of the feasible region.

Step 2. Evaluate f(λ) at its stationary points within the interior of the faces and edges of the feasible region identified in the constraint-vertex table..

Step 3. Evaluate f(λ) at the vertices of the feasible region.

Step 4. Select the global optimal solution(s).

Notice that it is easier to evaluate simply f(λ) at all stationary points and at the vertices and to select the largest or the smallest value, depending on whether the problem is of maximization or minimization type. Such an approach is much simpler than employing the needed the second order derivatives of f (λ) (or its Hessian) in deciding whether a given stationary point is a maximum or a minimum point.

Since the gradient is defined over an open set, we eliminate one of the appropriate λ_i's by utilizing $\Sigma\lambda_i = 1$ in making the domain an open set. Therefore, we solve unconstrained problems over some relevant open sub-domains of the feasible region. First, we find the critical points on the *interior* of the feasible region. An interior point of the feasible region is a point that satisfies all of the constraints, but the constraints are nonbinding at that point. Next, we find the critical points on the boundaries of the feasible region. Finally, evaluation of the objective function at critical points and vertices of the feasible region provides the global optimal solution. The removal of constraints in the proposed algorithm reduces the constrained optimization problem to some unconstrained optimization problems, which can be easily dealt with using the gradient.

5. Applications to Business Administration

Optimization is at the core of rational decision making in business. Even when the decision-maker has more than one goal, or when there is significant uncertainty in the system, optimization provides a rational framework for efficient decisions. The Markowitz mean-variance formulation is a classic example. This section illustrates the solution algorithm and its applications to finance, economics, marketing, and production in the context of numerical examples already solved by other methods.

Financial Example:

Although there are a variety of portfolio selection models, the widely used method is its formulation as a quadratic optimization problem.

The following portfolio selection is from Haugen (1997) that is solved by Karush-KuhnTucker conditions, which assumes that the objective function is convex. The information on the expected return vector (R) and covariance matrix (Cov.):

$$R = \begin{pmatrix} 0.05 \\ 0.10 \\ 0.15 \end{pmatrix} \qquad Cov = \begin{pmatrix} 0.20 & 0.15 & 0.17 \\ 0.15 & 0.21 & 0.09 \\ 0.17 & 0.09 & 0.28 \end{pmatrix}$$

As in Haugen (1997), we will use the quadratic performance measure as a minimization of risk:

$$\text{Min } f(\mathbf{X}) = 0.2X_1^2 + 0.21X_2^2 + 0.28X_3^2 + 0.3X_1X_2 + 0.34X_1X_3 + 0.18X_2X_3$$

subject to:

$$0.05X_1 + 0.1X_2 + 0.15X_3 = 0.1$$
$$X_1 + X_2 + X_3 = 1$$
$$X_1 , X_2 , X_3 \geq 0$$

Clearly, the objective function and equality conditions indicate this is a continuous optimization problem with a bounded feasible region. We now follow the algorithmic steps outlined in Section 4.

Step 1: For this three-dimensional decision-variable problem, the feasible region is the line segment joining the following two vertices:

Vertex	(X_1 , X_2, X_3)	f(X)
A	(0, 1, 0)	0.21
B	(1/2, 0, 1/2)	0.215

Step 2: Find the critical points on the line segment: The parametric representation of the line segment AB is:

$$X_1 = 0\lambda_1 + 1/2\lambda_2 = 1/2\lambda_2$$
$$X_2 = 1\lambda_1 + 0\lambda_2 = 1- \lambda_2$$
$$X_3 = 0\lambda_1 + 1/2\lambda_2 = 1/2\lambda_2 \ \forall\lambda_2 \in (0,1).$$

Using the chain-rule, the derivative of f(λ) with respect to λ_2 is:

$$f'(\lambda_2) = \Sigma\, \partial f(\mathbf{X})/\partial X_j \,.\, \partial X_j/\partial\lambda_2 =$$
$$(0.5X1 + 0.3X_2 + 0.34X_3)(1/2) +$$
$$(0.42X2 + 0.3X1 + 0.18X3)(-1) +$$
$$(0.56X3 + 0.34X1 + 0.18X2)(1/2) =$$
$$0.75\lambda_2 - 0.36,\ \lambda_2 \in (0,1)$$

The derivative vanishes at $\lambda_2 = 12/25$. This gives an interior critical point $C_1 = (X_1 = 1/2\lambda_2 = 0.24,\ X_2 = 1- \lambda 2 = 0/52,\ X_3 = 1/2\lambda_2 = 0.24)$ with objective values of 0.1668

Step 3: There is no edge for this problem.

Step 4: Now by comparing the numerical values of f(**X**) at vertices and point C_1, we conclude that the optimal solution is at point C_1 = (0.24, 0/52, 0.24) with optimal value 0.1668 .

By having the above information, one readily constructs the numerical tight bounds for the objective function, that is,

$0.1668 \le 0.2X_1^2 + 0.21X_2^{\ 2} + 0.28X_3^2 + 0.3X_1X_2 + 0.34X_1X_3 + 0.18X_2X_3 \le 0.215$, over its feasible region.

Non-Convex Quadratic Optimization:

Now suppose the covariance 0.34 is negative, say -0.5. The problem is:

Min $f(\mathbf{X}) = 0.2X_1^2 + 0.21X_2^{\ 2} + 0.28X_3^2 + 0.3X_1X_2 - 0.5X_1X_3 + 0.18X_2X_3$

subject to:

$$0.05X_1 + 0.1X_2 + 0.15X_3 = 0.1$$
$$X_1 + X_2 + X_3 = 1$$
$$X_1 , X_2 , X_3 \geq 0$$

Notice that the KKT condition-set does not provide any solution to this non-convex problem. However, with the new approach, we will follow the same steps as before.

Using the chain-rule, the derivative of $f(\lambda)$ with respect to λ_2 is:

$$f'(\lambda_2) = \sum_{j=1}^{3} \partial f(\mathbf{X})/\partial X_j \,.\, \partial X_j/\partial\lambda_2 =$$
$$(0.5X_1 + 0.3X_2 - 0.5X_3)(1/2) +$$
$$(0.42X_2 + 0.3X_1 + 0.18X_3)(-1) +$$
$$(0.56X_3 - 0.5X_1 + 0.18X_2)(1/2) =$$
$$0.045\lambda_2 + 0.18, \quad 0<\lambda_2<1$$

The derivative does not vanish. Therefore there is no interior stationary point for this problem.

Step 4: Comparing the numerical values of $f(\mathbf{X})$ at two vertices:

Vertex	(X_1 , X_2, X_3)	$f(\mathbf{X})$
A	(0, 1, 0)	0.21
B	(1/2, 0, 1/2)	0.01

we conclude that the optimal solution is at point B = (1/2, 0, 1/2) with optimal value 0.01.

By having the above information, one readily constructs the numerical tight bounds for the objective function, that is,

$$0.01 \le 0.2X_1^2 + 0.21X_2^2 + 0.28X_3^2 + 0.3X_1X_2 - 0.5X_1X_3 + 0.18X_2X_3 \le 0.21, \text{ over its feasible region.}$$

Economics Example:

The following Micro-Economics optimization with a logarithmic utility objective function is from Kreps (1990, pages 778-782) which is solved therein by applying the Karush-Kuhn-Tucker conditions. The problem involves maximizing the utility due to consumption of two products, wheat and corn. No more than $10 can be spent for the purchase of these products and the total caloric content may exceed 1500. The decision variables are: X_1 = number of units of wheat, and X_2 = number of units of candy.

Max $\quad f(\mathbf{X}) = 3\log(X_1) + 2\log(2 + X_2)$

subject to: $\quad X_1 + X_2 \le 10$

$150X_1 + 200X_2 \le 1500$

$X_1 \ge 0,\ X_2 \ge 0.$

In solving this problem, we follow the algorithmic steps outlined in Section 4.

Step 1: The graph of the feasible region provides the vertices, which are A, B, C, and D, respectively:

$X_1 = 10$	$X_1 = 9$	$X_1 = 0$	$X_1 = 0$
$X_2 = 0$	$X_2 = 1$	$X_2 = 31/9$	$X_2 = 0$

The parametric representation of the interior points of the feasible region is:

$X_1 = 10\lambda_1 + 9\lambda_2$

$X_2 = \lambda_2 + 31/4\lambda_3$, such that $1 > \lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$

The parametric objective function
(after the substitution $\lambda_4 = 1-\lambda_1 - \lambda_2 - \lambda_3$) is:

$$f(\lambda) = 3\log(10\lambda_1 + 9\lambda_2) + 2\log(\lambda_2 + 31/4\ \lambda_3 + 2)$$

Step 2: The gradient does not vanish anywhere, therefore, there is no any interior stationary point.

Step 3: Representation of f(X) on the edge joining the following two vertices B and C:

$X_1 = 9$ $X_1 = 0$
$X_2 = 1$ $X_2 = 31/4$

The parametric objective function (after substitution $\lambda_2 = 1 - \lambda_1$) is:

$$f(\lambda) = 3\log(9\lambda_1) + 2\log(39/4 - 27/4\lambda_1)$$

The derivative vanishes at $\lambda_1 = 13/15$. Therefore $\lambda_1 = 13/15$, $\lambda_2 = 2/15$ which gives $X_1 = 39/5$, $X_2 = 19/10$, with $f(\mathbf{X}) = 8.9$. On the other remaining edges the derivatives do not vanish.

Step 4: Evaluation of objective function at the vertices of the feasible region:

Therefore, the optimal solution occurs at $X_1 = 39/5$, and $X_2 = 19/10$

Vertex	(X_1, X_2)	$f(\mathbf{X})$
A	(10, 0)	3log10
B	(9, 1)	3log9
C	(0, 31/9)	2log(31/4)
D	(0, 0)	undefined

with an optimal value of 8.9.

By having the above information, one is able to provide only the numerical upper bound for the objective function, that is,

$3\log(X_1) + 2\log(2 + X_2) \leq 8.9$, over its feasible region.

Marketing Example:

The following Marketing nonlinear (quadratic) programming problem is from Markland and Sweigart (1987, pages 719-720) which is solved therein by the Lagrangian multiplier method. The problem involves maximizing the sales of a product subject to the total $1000 available for marketing that must be spent on newspaper and/or radio commercials advertising campaign. The decision variables are: X_1 = number of units of newspaper advertisements, and X_2 = number of units of radio commercials.

Max $\quad 4300X_1 - 300X_1^2 + 2500X_2 - 100X_2^2$

subject to: $\quad 300X_1 + 200X_2 = 1000$

$X_1 \geq 0, X_2 \geq 0$

We now follow the algorithmic steps outlined in Section 4.

Step 1: The vertices of the feasible region are:

$X_1 = 10/3 \qquad X_1 = 0$

$X_2 = 0 \quad X_2 = 5$

The parametric representation of the interior of the feasible region is:

$X_1 = 10/3\lambda_1$

$X_2 = 5\lambda_2$, such that $1 > \lambda_1, \lambda_2 > 0$, $\lambda_1 + \lambda_2 < 1$

After substituting for $\lambda_2 = 1 - \lambda_1$, we have:

$X_1 = 10/3\lambda_1$

$X_2 = 5 - 5\lambda_1$

Step 2: The parametric objective function is:

$$f(\lambda) = -17500/3\lambda_1^2 + 20500/3\lambda_1 - 10000$$

The derivative of $f(\lambda)$ vanishes at $\lambda_1 = 41/70$. Therefore $\lambda_1 = 41/70$, $\lambda_2 = 29/70$, and $f(\lambda) = 12000$. This gives $X_1 = 41/21$, and $X_2 = 29/14$, with $f(\mathbf{X}) = 12000$.

Step 3: Evaluation of objective function at the vertices of the feasible region:

Step 4: Therefore, the optimal solution occurs at $X_1 = 41/21$, and $X_2 = 29/14$ with an optimal value of 12000. By having the above information, one readily constructs the numerical tight bounds for the objective function, that is,

$2400 \leq 4300X_1 - 300X_1^2 + 2500X_2 - 100X_2^2 \leq 12000$, over its

Vertex	(X_1, X_2)	$f(\mathbf{X})$
A	(10/3, 0)	4000
B	(0, 5)	2400

feasible region.

Production and Operation Management Example:

Many practical nonlinear mathematical programming leads to optimizing a quotient of two functions subject to a set of linear constraints. The areas of applications include, cutting stock problem to minimize the ratio of wastage to useful output, ship scheduling to maximize the ratio of profit per journey to total journey, and the Markov chain decisions of finding minimum cost policies for the management of stochastic systems. The following fractional optimization problem is from Chadha (1999) that is solved by a specialized solution algorithm therein.

Max $\quad f(\mathbf{X}) = (2X_1 + 6X_2) / (X_1 + X_2 + 1)$

subject to:

$$X_1 + X_2 \leq 4,\ 3X_1 + X_2 \geq 6,\ X_1 - X_2 = 0,\ X_1, X_2 \geq 0$$

Clearly, the non-negativity conditions indicate that the denominator of the objective function does not vanish within the feasible region; therefore, this is a continuous optimization problem with bounded feasible region.We now follow the algorithmic steps outlined in Section 4.

Step 1: Since the feasible region is only a line segment, there are no interior critical points.

Step 2: Notice that, for this two-dimensional problem, the feasible region is the line segment joining the following two vertices:

Step 3: Finding the critical points on the line segment: The parametric representation of the line segment AB is:

$X_1 = 3/2\lambda1 + 2\lambda2 = 3/2\lambda1 + 2(1-\lambda1) = 2 - \lambda1/2$

Vertex	(X_1, X_2)	f(**X**)
A	(3/2, 3/2)	3
B	(2, 2)	16/5

$X_2 = 3/2\lambda1 + 2\lambda2 = 3/2\lambda1 + 2(1 - \lambda1) = 2 - \lambda1/2$

for all $0<\lambda1<1$.

Therefore, the parametric representation of the problem over AB is:

Max $f(\lambda1) = (16 - 4\lambda1)/(5 - \lambda1)$, over the domain $0< \lambda1 <1$.

The derivative of $f(\lambda1)$ is:

$f'(\lambda1) = -4/(5 - \lambda1)^2$

which is always negative over its domain. Therefore, there is no critical point for this problem.

Step 4: Now by evaluating f(X) at vertices, we conclude that the optimal solution is at (2, 2) with an optimal value of 16/5. This solution is superior to the solution (X1, X2) = (3/2, 3/2), with objective value of 3, obtained in the above reference.

By having the above information, one readily constructs the numerical tight bounds for the objective function, that is,

$3 \le (2X_1 + 6X_2) / (X_1 + X_2 + 1) \le 16/5$, over its feasible region.

6. Conclusi.ons

A critical evaluation of the current literature on even local optimization solution algorithms with business applications reveals that their approaches are frequently too complex, computationally laborious, and difficult to implement. Finding the global solution for general optimization problems is not an easy task. This paper proposes an enumeration procedure to obtain the global solution to linearly constrained optimization problems with almost differentiable objective functions. The key to the general solution algorithm is the removal of constraints through parametric representation of the problem using the convex combination of the vertices of its feasible region. The vertices are obtained by a simple and direct algebraic method. The globally optimal solution is then found by computing stationary points and evaluating the objective function at these points as well as at the vertices. The algorithm and its applications are presented in the context of small numerical problems.

This paper presented a new solution algorithm for linearly constrained optimization problems. Other more complicated approaches often yield solutions that are only locally optimal, although there may be no clear signal that it is not a globally optimal solution. The simple algorithm presented in this paper guarantees that the solution is globally optimal. The key to the general solution algorithm is the removal of constraints through parametric representation of the problem using the vertices of its feasible region. The vertices can be obtained by a simple and direct algebraic method. The globally optimal solution is then found by computing stationary points and evaluating the objective function at these points, as well as at the vertices. The algorithm and its applications were presented in the context of business decision models. This enables us to solve a large class of problems including linear, fractional, quadratic, convex, and non-convex programs.

Linear dependence among the linear constraints is commonplace in practical problems. Linear dependence among linear constraints that are binding at a point is commonly known as *degeneracy*, and such a point is called a *degenerate point*. The resolution of *degeneracy at a vertex* is essentially a combinatorial problem whose solution may require a significant amount of computation; see e.g. Gill et al (1989), or Yamada et al (1994). Since our approach is an explicit enumeration technique, as opposed to a path-following method, it is free from computational problems that may be caused by degeneracy.

The new algorithm compares favorably for small size problems with other methods, in that it has simplicity, potential for wide adaptation, and deals with all cases. The overhead involved is generating the vertices of the feasible region, finding the critical points, and performing some functional evaluations.

The algorithm is applicable to a wide variety of problems in finance, economics, marketing, production management and other business disciplines. Because of its simplicity, it has exceptional merit as a pedagogical tool in business programs, and it can be easily understood and applied by business managers without requiring advanced training in optimization techniques.

While other derivative-based algorithms, such as Lagrangian multiplier and Karush-Kuhn-Tucker (KKT) conditions, give locally optimal solutions, our solution algorithm provides the globally optimal solution in all cases. Notice that both the Lagrangian and KKT methods remove the constraints by using a linear combination of the constraints. Our solution algorithm uses the linear convex combination of the vertices of the feasible region to remove the constraints.

Notice also that, the computational comparison of the proposed algorithm with other methods is not likely to be very meaningful, since other general methods do not provide exact globally optimal solutions in all cases. This does not imply that all distinctions among specific classes of the problem should be ignored, since incorporating special characteristics of a problem to modify a general solution algorithm is always likely to result in increased computational efficiency. This is an area for future research. The proposed algorithm has the following features:

- It guarantees locating a global solution.

- It has simplicity in that it is easy to understand, and uses only numerical functional evaluation and the first order derivatives (gradient).

It can thus serve as a useful introduction to nonlinear optimization before covering advanced techniques.

Further areas of research include the extension of the solution algorithm to the special cases, such as unbounded feasible region, sensitivity analysis, Arsham (1998), and development of a computerized version with some possible refinements for large-scale problem implementation. At this point, the reader is asked to solve his/her own problems by applying this algorithm as a final measure of evaluation.

References

Arsham, H. (1998), Managing Cost Uncertainties in Transportation and Assignment Problems, *Journal of Applied Mathematics &*

Decision Sciences, Volume 2, (1), pp. 65-102.

Arsham, H. (1997), Affine Geometric Method for Linear Programs, *Journal of Scientific Computing*, Volume 12, (3), pp. 289-303.

Arsham, H. (1995), A Comprehensive Simplex-Like Algorithm for Network Optimization and Perturbation Analysis, *Optimization*, Volume 32, (3), pp. 211-267.

Arsham, H. (1993), Managing Project Activity-Duration Uncertainties, *Omega: International Journal of Management Science*, Volume 21, (2), pp. 111-122.

Arsham, H. (1992), Postoptimality Analysis for the Transportation Problem, *Journal of the Operational Research Society*, Volume 43, (1), pp. 121-139.

Arsham, H. (1990), A Complete Algorithm for Linear Fractional Programs, *Computers & Mathematics with Applications*, Volume 20, (2), pp. 11-23.

Arsham, H., and Stanton K. (2003), *Teaching Optimization at the MBA level*, Working Paper, University of Baltimore, Baltimore, MD, USA.

Barrientos, O., and Correa, R. (2000), An Algorithm for Global Minimization of Linearly Constrained Quadratic Functions, *Journal of Global Optimization*, Volume 16, (1), pp. 77-93.

Chadha, S. (1999), A Linear Fractional Program with Homogenous Constraints, *Opsearch*, Volume 36, (4), pp. 390-398.

Elton, E., Gruber, M., Brown, S., and Goetzman, W. (2003), *Modern Portfolio Theory and Investment Analysis*, John Wiley and Sons, Inc., New York, pp. 182-208.

Gill, P., Murray, W., Saunders, M., and Wright, M., (1989), A PracticalAnticycling Procedure for Linearly Constrained Optimization, Mathematical Programming, Volume 45, (3), pp. 437-474.

Haugen, R. (1997), *Modern Investment Theory*, Fourth edition, Prentice Hall, Upper Saddle River, New Jersey, pp. 92-130.

Hillier F., and Lieberman G. (1995), *Introduction to Operations Research*, McGrawHill Co., New York.

Hiriart-Urruty, J. (1998), Conditions for Global Optimality, *Journal of Global Optimization*, Volume 13, (3), pp. 349-367.

Horst, R., Nast, M., and Thoai, N. (1995), New LP-Bound in Multivariate Lipschitz Optimization: Theory and Applications, *Journal of Optimization Theory and Applications*, Volume 86, (2), pp. 369-388.

Horst, R., and Tuy, H. (1996), *Global Optimization, Deterministic Approaches*, Springer, Berlin.

Kreps, D. (1990), *A Course in Microeconomic Theory*, Princeton University Press, New Jersey.

Markland, R. and Sweigart, J. (1987), *Quantitative Methods: Applications to Managerial Decision Making*, John Wiley & Sons, New York.

Markowitz, H.(1959), *Portfolio Selection Efficient Diversification of Investments*, John Wiley and Sons, New York.

Sharpe W., Alexander, G., and Bailey, J. (1999), *Investments*, Prentice Hall, Upper Saddle River, New Jersey, pp. 193-200.

Thi H.., D. Pham, and Thoai N. (2002), Combination Between Global and Local Methods for Solving an Optimization Problem Over the Efficient Set, *European Journal of Operational Research*, Volume 142, (2), pp. 258-270.

Yamada, T., Yoruzuya, J., Kataoka, S (1994), Enumerating Extreme Points of a Highly Degenerate Polytope, *Computers and Operations Research*, Volume 21, (3), pp. 397-410.