**How Graphical Method for LP Sensitivity Range Can Go Wrong?**

**An Algebraic Approach Based on the Optimal Solution of Either the Dual OR the Primal.**

**A Critic**, i.e., we are interested in finding the **Scopes** and **Limitations** of Textbook pages 80-93.

**The Graphical Method: Scopes and Its Limitations**:

* It ignores sensitivity of the RHS for any non-binding constraints
* When it works correctly, it is limited to two-dimensional problems
* It may not work for some problems, as the following counter example can show:

**A Counterexample**:

Maximize 5X1 + 3X2

X1 + X2 ≤ 2

X1 – X2 ≤ 0

X1 ≥ 0

X2 ≥ 0.

It gives **Incorrect** Sensitivity Range for Coefficients of Objective Function:

-1 ≤ - c1 / c2 ≤ 1

**A General Algebraic Method As An Alternative Approach**:

We describe the new approach in the context of a numerical example:

**Numerical Example**: As our numerical example for RHS changes consider the carpenter problem:

Maximize 5 X1 + 3 X2

Subject to:

2 X1 + X2 ≤ 40 labor constraint

X1 + 2 X2 ≤ 50 material constraint

and both X1, X2 are non-negative.

The Optimal solution is X1 = 10, and X2 = 20. At the optimal vertex both the first two constrains are Binding, we w interested with Sensitivity Range of the first constraint.

**1. Sensitivity Range for Right-hand-sides**

**Sensitivity Range for RHS1**

Consider, the RHS1 parametric version of this problem:

Maximize 5 X1 + 3 X2

Subject to:

2 X1 + X2 = 40 + r1

X1 + 2 X2 = 50

and both X1, X2 are non-negative.

Page 1/8

The parametric solution is:

X1 = 10 + 2r1/3

X2 = 20-r1/3

Notice that the parametric solution contains the current optimal solution, by setting r1 = 0.

This solution must satisfy all other constraints:

X1 ≥ 0, X2 ≥ 0 substituting for X1 and X2, we have:

10 + 2r1/3 ≥ 0, 20 -r1/3 ≥ 0

This gives the range:

r1 ≥ - 15 **Interpretation**: The maximum amount of change (decrease) on the RHS1 for which Its Shadow Prices (7/3) remain unchanged,

r1 ≤ 60 **Interpretation**: The maximum amount of change (increase on the RHS1 for which It Shadow Price (7/3) remain unchanged.

*Notice that with parametric solution,*

*X1 = 10 + 2r1/3*

*X2 = 20-r1/3*

*To check we see that the optimal solution is (X1 = 10, and X2 = 20), for the unperturbed problem. Further more one obtains the parametric Optimal Value of:*

*5 X1 + 3 X2 = 5 (10 + 2r1/3) + 3 (20-r1/ ) = 110 + 7/3 r1*

*As you see, the optimal value for the primal is 110, and the* ***Shadow Price*** *of RH1 is 7/3,*

*The Rate of Change in optimal value with respect to change in the first RHS.*

Similarly one obtains for **Sensitivity Range for RHS2**

2 X1 + X2 = 40

X1 + 2 X2 = 50 + r2

and both X1, X2 are non-negative. The parametric solution is:

X1 = 10 - r2/3

X2 = 20 + 2r2/3

This solution must satisfy all other constraints:

X1 ≥ 0, X2 ≥ 0 substituting for X1 and X2, we have:

10 - r2/3 ≥ 0, 20 + 2r2/3 ≥ 0. This gives,

Page 2/8

r2 ≤ 30 **Interpretation**: The maximum amount of change (increase) on the RHS2 for which It Shadow Price (1/3) remain unchanged.

r2 ≥ -30 **Interpretation**: The maximum amount of change (decrease) on the RHS2 for which It Shadow Prices (1/3) remain unchanged.

*Notice that with parametric solution,*

*X1 = 10 -r2/3*

*X2 = 20 + 2r2/3*

*To check we see that the optimal solution is (X1 = 10, and X2 = 20), for the unperturbed problem contains in this solution. Further more one obtains the parametric Optimal Value of:*

*5 X1 + 3 X2 = 5 (10 – r2/3 ) + 3 ( 20+2r2/3 ) = 110 + 1/3 r2*

*As you see, the optimal value for the primal is 110, and the* ***Shadow Price*** *of RH1 is 1/3,*

*The Rate of Change in optimal value with respect to change in the first RHS.*

 ***Therefore the solution to the Dual Problem is (U1= 7/3, and U2 = 1/3)***

**2. Sensitivity Analysis of Objective Function Coefficients**

The Dual Problem is:

Minimize 40 U1 + 50 U2

Subject to:

2U1 + 1U2 ≥ 5 Net Income from a table

1U1 + 2U2 ≥ 3 Net Income from a chair

and U1, U2 are non-negative.

The Optimal solution is U1 = 7/3, and U2 = 1/3, as found earlier. At the optimal vertex both the first two constrains are Binding, we w interested with Sensitivity Range of the first constraint.

**Sensitivity Range for C1 = 5**

Consider, the RHS1 parametric version of this problem:

2U1 + 1U2 = 5 + c1

1U1 + 2U2 = 3

The parametric solution is:

U1 = 7/3 + 2c1/3

U2 = 1/3 -c1/3

This solution must satisfy all other constraints:

U1 ≥ 0, U2 ≥ 0 substituting for U1 and U2, we have:

Page 3/8

1/3 -c1/3 ≥ 0

7/3 + 2c1/3 ≥ 0

This gives the range:

c1 ≤ 1 **Interpretation**: The maximum amount of change (increase) on the first cost coefficient 1, for which It Optimal Solution (X1 = 10) remains unchanged.

c1 ≥ -7/2 **Interpretation**: The maximum amount of change (decrease) on the first cost coefficient 1, for which It Optimal Solution (X1 = 10) remains unchanged.

**Sensitivity Range for C2 = 3**

Consider, the RHS2 parametric version of this problem:

2U1 + 1U2 = 5

1U1 + 2U2 = 3 + c2

and U1, U2 are non-negative.

U1 = 7/3 - c2/3

U2 = 1/3 +2c2/3

This solution must satisfy all other constraints:

U1 ≥ 0, U2 ≥ 0 substituting for U1 and U2, we have:

7/3 - c2/3 ≥ 0

1/3 +2c2/3 ≥ 0

This gives the range:

c2 ≥ -1/2, c2 ≤ 7

c2 ≥ -1/2 **Interpretation**: The maximum amount of change (decrease) on the first cost coefficient 2, for which It Optimal Solution (X2 =20) remains unchanged,

c2 ≤ 7 **Interpretation**: The maximum amount of change (increase) on the first cost coefficient 1, for which It Optimal Solution (X2 =20) remains unchanged,

*One obtains the parametric Optimal Value of:*

*40U1 + 50U2 = 40(*7/3 - c2/3*) + 50(1*/3 +2c2/3*) = 110 + 20c2*

*As you see, the optimal value for the dual is 110 (as expected), and the* ***Shadow Price*** *of the dual (i.e., solution for the primal is (X2 =20).*

Page 4/8

**LINDO OUTPUT:**

Max 5X1 + 3X2

S.T. 2X1 + X2 <40

X1 + 2X2 <50

End

 OBJECTIVE FUNCTION VALUE

 1) 110.0000

 VARIABLE VALUE REDUCED COST

 X1 **10.000000** 0.000000

 X2 **20.000000** 0.000000

 ROW SLACK OR SURPLUS DUAL PRICES

 2) 0.000000 **2.333333**

 3) 0.000000 **0.333333**

**(Corrected) The Range for the Coefficients of Objective Function for which the Optimal Solution Remains Unchanged**

 **OBJ COEFFICIENT RANGES**

 VARIABLE CURRENT ALLOWABLE ALLOWABLE

 COEF INCREASE DECREASE

 X1 5.000000 **1.000000 3.500000**

 X2 3.000000 **7.000000 0.500000**

 RIGHTHAND SIDE RANGES

**(Corrected) The Range for the RHS values for which the Shadow Prices (solution to the Dual Problem) Remains Unchanged**

  **RHS RANGES**

 ROW CURRENT ALLOWABLE ALLOWABLE

 RHS INCREASE DECREASE

 2 40.000000 **60.000000 15.000000**

 3 50.000000 **30.000000 30.000000**

Notice That Each LINDO Report is a **Combined** (i.e., for Primal and for the Dual Problems) **Report**.

Page 5/8 continues…..

Min 40U1 + 50U2

S.T.

2U1 + U2> 5

U1 + 2U2 > 3

End

 OBJECTIVE FUNCTION VALUE

 1) 110.0000

 VARIABLE VALUE REDUCED COST

 U1 **2.333333** 0.000000

 U2 **0.333333** 0.000000

 ROW SLACK OR SURPLUS DUAL PRICES

 2) 0.000000 **-10.000000**

 3) 0.000000 **-20.000000**

 **OBJ COEFFICIENT RANGES**

**(Corrected) The Range for the Coefficients of Objective Function for which the Optimal Solution Remains Unchanged**

 VARIABLE CURRENT ALLOWABLE ALLOWABLE

 COEF INCREASE DECREASE

 U1 40.000000 **60.000000 15.000000**

 U2 50.000000 **30.000000 30.000000**

 **RIGHTHAND SIDE RANGES**

**(Corrected) The Range for the RHS values for which the Shadow Prices (solution to the Dual Problem) Remains Unchanged**

 ROW CURRENT ALLOWABLE ALLOWABLE

 RHS INCREASE DECREASE

 2 5.000000 **1.000000 3.500000**

 3 3.000000 **7.000000 0.500000**

Notice That Each LINDO Report is a **Combined** (i.e., for Primal and for the Dual Problems) **Report**.

Page 5/8 continues…..

Changing Minimization to Maximization

Max -40U1 -50U2

S.T. 2U1 + U2 >5

U1 + 2U2 >3

End

OBJECTIVE FUNCTION VALUE

1. -110.0000

 VARIABLE VALUE REDUCED COST

 U1 2.333333 0.000000

 U2 0.333333 0.000000

 ROW SLACK OR SURPLUS DUAL PRICES

 2) 0.000000 -10.000000

 3) 0.000000 -20.00000

 OBJ COEFFICIENT RANGES

 VARIABLE CURRENT ALLOWABLE ALLOWABLE

 COEF INCREASE DECREASE

 U1 -40.000000 15.000000 60.000000

 U2 -50.000000 30.000000 30.000000

 RIGHTHAND SIDE RANGES

 ROW CURRENT ALLOWABLE ALLOWABLE

 RHS INCREASE DECREASE

 2 5.000000 1.000000 3.500000

 3 3.000000 7.000000 0.500000

**Some Other Interesting Managerial Sensitivity Analysis Activities:**

Consider the following LP problem

Max 40X1 + 50X2

S.T.

X1 +2x2 ≤ 40

4X1 + 3X2 ≤ 120

X1 ≥ 0, X2 ≥ 0

With given the following information:

The optimal solution is (X1 = 24, X2 = 8)

The shadow prices of (U1 = 16, U2 = 6)

**1. Suppose the first constraint is changed to**

1.33X1 +2x2 ≤ 40

What is the impact on optimal solution and optimal value?

Since the first constraint is a binding one, one has to **re-solve** the problem to find the new optimal solution and optimal value.

**2. Suppose the following new constraint is added:**

0.2X1 +0.1x2 ≤ 5

What is the impact on optimal solution and optimal value?

The current optimal solution (X1 = 24, X2 = 8)

0.2(24) +0.1(8) = 5.6

The current solution does not satisfy the new constraint. One must add the new constraint and resolve the problem with three constraints.

**3. Suppose a new product takes 1.2 and 2 units of resources one and two respectively, it brings $30 in net profit. Should we produce the new product or not?**

Cost of producing the new product is:

1.2(16) + 2(6) =$31.2

Since it is not profitable, do not produce.

Page 6/8

**Corrections for the Graphical Method That Always Works**

**The Carpenter's Problem:**

Maximize 5X1 + 3X2

Subject to:

2X1 + X2  40

X1 + 2X2  50, X1  0, X2  0

Computation of allowable increase/decrease on the C1 = 5: The binding constraints are the first and the second one. Perturbing this cost coefficient by c1, we have 5 + c1, using slopes proportionality, we have:

(5 + c1)/2 = 3/1, for the first constraint, and (5 + c1)/1 = 3/2 for the second constraint. Solving these two equations, we have: c1 = 1 and c1 = -3.5. The allowable increase is 1, while the allowable decrease is 1.5. As far as the first cost coefficient C1 remains within the interval [5 - 3.5, 5 + 1] = [1.5, 6], the current optimal solution remains.

Similarly for the second cost coefficient C2 = 3, we have the sensitivity range of [2.5, 10].

As another example, consider the earlier problem:

Maximize 5X1 + 3X2

Subject to:

X1 + X2  2

X1 - X2  0

X1  0

X2  0

Computation of allowable increase/decrease on the C1 = 5: The binding constraints are the first and the second one. Perturbing this cost coefficient by c1, we have 5 + c1, using slopes proportionality, we have:

(5 + c1)/1 = 3/1, for the first constraint and (5 + c1)/1 = 3/(-1) for the second constraint. Solving these two equations, we have: c1 = -2 and c1 = -8. The allowable decrease is 2, while the allowable increase is unlimited. As far as the first cost coefficient C1 remains within the interval [ 5 - 2, 5 + ] = [3, ], the current optimal solution remains optimal.

Similarly, for the second cost coefficient C2 = 3 we have the sensitivity range of [3 - 8, 3 + 2] = [-5, 5].

Page7/8

**Notice that **Means:

- It does not exists, i.e., (because it is not a number, it is a concept, only)

- The upper bound is unlimited, i.e.,

- It is the Big-M, i.e., the largest positive number you can imagine.

**Finding the Sensitivity Range of the nonbinding constraints:**

Consider the following LP problem

 Max 7X1+10X2

Subject to:

5X1 + 6X2 ≤ 3600 (Raw material)

X1 + 2X2 ≤ 960 (Labor)

X1 ≤ 500 (Production limit)

X2 ≤ 500 (Production limit)

X1, X2 ≥ 0 (Non-negativity)

With optimal solution of X1 = 360, X2 = 300, we wish to find sensitivity rang for the first non-binding constraint X1 ≤ 500.

The parametric RHS3 at its binding position is X1 = 500 + c3. Plugging the current solution we have 360 = 500 + c3. This gives c3 = -140, therefore one can decrease the RHS3 by 140, the amount of increase is unlimited. Similarly, one can find the sensitivity range for the fourth constraint, RHS4: X2 = 500 + c4. Plugging the current solution we have 300 = 500 + c4. This gives c4 = -200, therefore one can decrease the RHS4 by 200, the amount of increase is unlimited.

Page 8/8