

Management Science, Logistics, and Operations Research

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The nature of emerging business technologies, such as automatic identification, and data capture innovations, such as smart cards, touch memory, and RFID, proves to be a difficult process to implement and achieve, despite recent efforts, even as implementation continues to lessen. The economy of scales associated with bar codes is a difficult barrier to overcome. Two Pittsburgh-based companies are showcased in this case study, namely Mobile Aspects, Inc., a recognized leader in the integration of RFID-related technologies in inventory management processes of large acute care, and Vocollect, Inc., which is especially noted for its voice-recognition software and small hardware platforms used in warehousing and partially automated inventory systems. The firms' goals for implementing inventory management, specific inventory recommendations and changes, comparison of inventory management processes, selected measures to ensure the quality and security of data transmitted via RFID-based technologies, and lessons learned are discussed.

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An Integer Programming (IP) model is formulated and optimally solved for a real-life production planning problem. The model is used to determine the optimum production plan for the formation stage of battery manufacturing in a local battery-producing company. Battery manufacturing is a complicated process that involves several stages and several chemical and physical operations. The formation stage is considered the bottleneck of this process because it has the most critical limitations of time and production resources. In the formation stage, batteries are filled with acid and charged with electricity using different types of circuits. The objective of the model is to maximize the productivity of the critical formation stage by allowing the best utilization of the limited time and equipment resources. The model is able to consider a large number of battery sizes and types and also a large number of charging circuits with different capacities and charging speeds. The model-generated optimum production plans increase daily profits by 12% on average in comparison to manually-generated production plans.

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Multimodal supply chains are characterized by multiple changes of transport modes, vehicles, and transport operators, and hence, the risks for theft, untimely delivery, and freight quality deterioration increase. There is a growing need to manage the security and efficiency of consignments. Tracking and tracing services, which are on the market, mainly concentrate on a single mode of transport, such as road vehicles, or transport units, such as containers. This chapter describes a concept for managing the security of multimodal supply chains using monitoring technology. A profound risk analysis was performed to identify threats and vulnerabilities during the different phases of the supply chain and key threat scenarios. A security service concept was developed to cope with the risks using monitoring technology. The technological solution depends on the needs of the supply chain actors and on the supply chain properties. The service concept was tested by monitoring different international shipments leaving from Finland. Intelligent monitoring devices, which were attached to the consignments, collected information on location and environmental parameters and transmitted in real-time to a background system. The chapter gives an overview of the framework, service concept, and the analysed results of multimodal shipments.

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The use of RFID (Radio Frequency Identification) tracking in closed systems is replacing barcode as dominant tracking system in many industrial sectors because RFID enables reading multiple objects simultaneously without visual contact. To enlarge tracking systems to cover global supply chains, all aspects related to RFID, such as radio frequencies, data content, transmission protocols, and message sets, need to be standardised. By collecting, processing, and distributing information efficiently, organisations should be able to improve the efficiency of their transport logistics processes, lower their operational costs, and improve their portfolio of logistics services. This chapter describes the current perspectives, challenges, and benefits of RFID tracking applications in manufacturing industry. The perspectives derived from review of previous research are validated by using case study method.

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While developing the inventory model with shortages under permissible delay in payments, it has been observed in the literature that the researchers have not considered the fact that the retailer can earn interest on the revenue generated after fulfilling the outstanding demand as soon as he receives the new consignment at the start of the cycle. Owing to this fact, the present study investigates the impact of interest earned from revenue generated after fulfilling the stock out at the start of the cycle on a single commodity inventory model with shortages for deteriorating item, in which the whole lot goes through an inspection process on arrival before entering into inventory system, under the conditions of permissible delay in payments. After inspection, the non-defective items are retained to fulfill the demand and the defective items are returned to the supplier. The results have been demonstrated with the help of a numerical example using the tools of Matlab7.0.1.

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We can address innovation from different perspectives. In engineering practices we can look to changes resulting from attempts to discover ways of overcoming difficulties. How can we manage these innovative practices in engineering design projects? Furthering our perspective we use an actor-network way to look at change processes as chains of translations between heterogeneous actors that are enrolled in changes and where patterns of action are inscribed in durable manners. In an actor-network, the chains of inscriptions are stronger if the number of aligned actors is bigger, and this is the case of engineering projects where the cost of change builds up with time. Through the use of some stylized situations, the authors construct a Perturbation Index to obtain numeric values to assess the dynamics of innovations in engineering practices. The aim is that the application of this index to real situations could lead to meaningful descriptions of such innovation processes. Managing innovation in engineering design projects has to do with the management of project scope. The proposal extends scope management from its definition and planning phases through the control of changes along the execution.

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This chapter provides a critical overview of Linear Programming (LP) from a manager's perspective. The main objective is to provide managers with the essentials of LP as well as cautionary notes and defenses on common modeling issues and software limitations. The authors illustrate the findings by solving a simple LP directly on the original decision variables and constraints space without adding new variables or translating the model to fit a specific solution algorithm. The aims are the unification of diverse set of topics in their natural states in a manner that are easy to understand and providing useful information to the managers. The advances in computing software have brought LP tools to the desktop for a variety of applications to support managerial decision-making. However, it is already recognized that current LP tools, in ample circumstances, do not answer the managerial questions satisfactorily. For instance, there is a costly difference between the mathematical and managerial interpretations of sensitivity analysis. LP software packages provide one-change-at-a-time sensitivity results; the authors develop the largest sensitivity region, which allows for simultaneous dependent and/or independent changes, based on the optimal solution. The procedures are illustrated by numerical examples including LP in standard-form and LP in non standard-form.

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to integrate the people, technology, and processes in place; and 4) that people engagement hinges upon six factors (trust, beliefs, communications, culture, reward systems/metrics, and synergy). This chapter is a modification of an earlier work published by IGI Global.

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Industrialization leaves no doubt that our planet is suffering from global warming, depletion of natural resources, pollution, waste, and other environmental concerns. Consequently, businesses, communities, and governments have been environmentally conscious and shown growing concern for sustainable development, particularly following the establishment of the United Nations Environmental Program (UNEP). This chapter provides further insights into sustainability and supply chain research through adopting a stakeholder perspective to understand drivers and consequences of supply chain sustainability in the United Arab Emirates. The results indicate that pressures enforced by champions and customers are positively related to supply chain sustainability. However, the relationship between government pressures and supply chain sustainability is insignificant. Moreover, the findings provide evidence that the impact of supply chain sustainability on the organizational sustainable performance is significantly positive. Supply managers can learn from these results in developing sustainable initiatives earlier along their supply chain through selecting and evaluating suppliers based on sustainability-related standards. In addition, environmental collaboration with customers and suppliers based on knowledge sharing and application may identify and reduce the total environmental impact.

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Ronald John Lofaro, Embry-Riddle Aeronautical University, USA

It is well over 30 years since the first (then called) Cockpit Resource Management (CRM) training, now called crew resource management was introduced. It is a shibboleth, a sacred cow as it were, despite many issues, concerns, and changes over the years. Some 21 years ago, 1992, an Air Transport Association (ATA)/Federal Aviation Association (FAA)-Sponsored Workshop was convened in an attempt to deal with some specific CRM issues. Yet the issues and needs as articulated in that workshop, and some

newer ones, remain. Thus, this chapter is 21 years overdue, leading to the questions: Why now and is it still relevant? As said, some needs, issues, and concerns remain. The relevancy is that both a critique of civil aviation CRM on many levels and a comparison with current USAF, USCG, and USN CRM are presented. The proposed skeletal template for the long-overdue revision of civil aviation CRM, the R-MPM is shown. Next, a new model for an intelligent cockpit automated decision aid/advisory system, Event Response Integrated Decision Advisories (ERICA), is shown. ERICA came about from 2009-2012 work in automated decision-making tools for the cockpit and the realization that the Revised Mission Performance Model (R-MPM) and ERICA were interrelated.

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Jesús Alcoba González, La Salle Campus Madrid, Spain

Highly industrialized services add value to a company, but at the same time, they can destroy it in the case of customers, if it happens that an almost identical proposal is made to two different clients who are seeking different experiences. The analysis of human subjectivity shows that the interaction with services goes beyond its mere use, as it is related to human beings' search for meaning, and it can potentially become a part of its users' biography and identity. The present chapter compiles some of the models that can contribute to getting over the paradox of service industrialization and which are here divided into two types: those that allow for a deep knowledge of the customer and those that are based upon designing experiences as a value proposition. The present chapter also suggests a research agenda that aims to get over the paradox of service industrialization.

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Kwok Hung Lau, Royal Melbourne Institute of Technology University, Australia

Haibo Huang, Royal Melbourne Institute of Technology University, Australia

This paper reports the findings of a questionnaire survey on the adoption and utilization of information and communication technology (ICT) – an important tool to enhance efficiency and responsiveness in modern-day supply chains – in the Chinese third-party logistics (3PL) industry. As a global manufacturing base, China has a burgeoning 3PL industry serving a large number of domestic firms as well as multinational corporations. With more and more organizations outsourcing their logistics function to 3PL firms, the latter have to make use of ICT for efficient communication with clients and coordination of activities in order to work hand-in-hand with customers to meet their day-to-day logistics needs. To investigate the status quo of ICT utilization in China, a questionnaire survey was conducted in 2009 to investigate the level of ICT adoption by the 3PL firms. For comparison, the findings of the Annual Third-Party Logistics Studies from 2007 to 2013 in this regard were also analyzed. The questionnaire survey results reveal that ICT is being widely adopted in the Chinese 3PL industry suggesting a high level of awareness of its significance and benefits to both the service providers and the clients. The ICT used ranges from low-tech telephone, facsimile, Internet access, to more advanced and sophisticated radio frequency identification technology and enterprise resource planning system. While small 3PL companies are using less expensive ICT at a tactical level mainly to cut costs and reduce errors in the day-to-day logistics operations, medium-sized and large firms have evolved to make use of more expensive ICT for planning and strategic purposes, such as business control, customer integration, and service differentiation. Transportation management system is regarded as the most important IT for business to most 3PL firms surveyed whereas the significance of other systems for warehouse management, order management, and inventory management, etc. varies depending on the services provided and the resources available. A follow-up review of the recent literature suggests that ICT adoption in the 3PL industry of China has not significantly increased since the 2009 survey but the general awareness of the importance of ICT capabilities is growing.

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Isabelle Mirbel, University of Nice Sophia-Antipolis, France

Electronic collaboration in the field of Supply Chain Management notably allows enhanced visibility for each of the involved actors through information sharing and global optimization through coordination. The proper achievement of these advantages is nevertheless not trivial since it implies the development of rather complex IT systems. Each actor is indeed mostly dealing with his own data semantics and custom processes so that data centralization and sharing is seldom achieved. Moreover, even if there would be a willingness to share data, this inherent heterogeneity would constitute a serious burden for software developments. The present chapter presents the results of a research aimed to partially solve these issues. It indeed presents the analysis and design of a platform for e-collaboration among the main Outbound Logistic (OL) actors based on a common conceptualization (including unified data semantics) and a set of services (supporting business processes realization). The chapter briefly presents the generic processes but mostly focuses on the representation of these services at a strategic level through an analysis of their added value and risk for OL actors potentially adopting the software solution.

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Angela Lin, University of Sheffield, UK

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In search of cheaper resources and being able to serve nearby market more efficiently in order to maintain and improve their market position in a global economy, companies relocate, extend, or establish new production facilities overseas. Because of this, how to manage the Information Systems (IS) that support transitional activities within companies has become an important agenda not only in the business but also in the academic community. At a glance, replicating information systems and value chain that a company already has in the new facility overseas is seemingly a cheap and easy option. Nevertheless, in practice, it is not a trivial task, and at its worst, the IS can become obsolete. This chapter explains why replication strategy does not always work, and drawing on the practice lens approach, it posits that any changes in technology-in-use are the outcome of an ongoing structuring process where people constitute and reconstitute the structure of an IS in use. The chapter demonstrates that the structure of an IS is not defined a priori but emerges from daily use of the system, along with people's understandings of the system, and of its role in the context of organisational routines. On this view, any challenges arising from the attempt to replicate the IS can be regarded as largely inevitable, since the structures embedded and enacted in the use of system, along with the practices that in turn recursively structure the use of the system would also have changed.

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In 2010, a framework aiming to address strategic investment decisions on IT infrastructure was developed. It was based in Benefits Management principles and Enterprise Architecture concepts, being inspired by the emerging public cloud technological trend. Meanwhile, the public cloud concept did not materialise at the expected pace and other alternatives have emerged in the market, in particular the private cloud-based solutions. This fact required the framework to be updated to cope with the business

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With the evolution of information technology, firms offshore outsource services to developing and low service cost countries to have cost as well competitive advantages. This is a growing practice, but there has been limited empirical attention in understanding the outsourcing phenomenon, particularly from the perspective of service provider firms that execute important business processes for their overseas clients. This shows the need to study the factors that play a significant role in the growing trend to outsource and why only a few service provider firms report success. In this chapter, the authors try to find factors that influence performance of service provider firms. Multiple regressions using four indicators of firm performance are carried out to see the influence of certain factors on Information Technology Enabled Service (ITES) firms’ performance.

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Michael J. Gravier, Bryant University, USA
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Increasingly, research across many disciplines has recognized the shortcomings of the traditional “integration prescription” for inter-organizational knowledge management. This research conducts several simulation experiments to study the effects of different rates of product change, different demand environments, and different economies of scale on the level of integration between firms at different levels in the supply chain. The underlying paradigm shifts from a static, steady state view to a dynamic, complex adaptive systems and knowledge-based view of supply chain networks. Several research propositions are presented that use the role of knowledge in the supply chain to provide predictive power for how supply chain collaborations or integration should evolve. Suggestions and implications are suggested for managerial and research purposes.

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Chapter 7

Essentials of Linear Programming for Managers: From System of Inequalities to Software Implementation

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ABSTRACT

This chapter provides a critical overview of Linear Programming (LP) from a manager's perspective. The main objective is to provide managers with the essentials of LP as well as cautionary notes and defenses on common modeling issues and software limitations. The authors illustrate the findings by solving a simple LP directly on the original decision variables and constraints space without adding new variables or translating the model to fit a specific solution algorithm. The aims are the unification of diverse set of topics in their natural states in a manner that are easy to understand and providing useful information to the managers. The advances in computing software have brought LP tools to the desktop for a variety of applications to support managerial decision-making. However, it is already recognized that current LP tools, in ample circumstances, do not answer the managerial questions satisfactorily. For instance, there is a costly difference between the mathematical and managerial interpretations of sensitivity analysis. LP software packages provide one-change-at-a-time sensitivity results; the authors develop the largest sensitivity region, which allows for simultaneous dependent and/or independent changes, based on the optimal solution. The procedures are illustrated by numerical examples including LP in standard-form and LP in non standard-form.

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1. INTRODUCTION

Linear programming (LP) has been a fundamental topic in the development of managerial decision-making. The subject has its origins in the early work of L. B. J. Fourier (1820s) on attempting to solve systems of linear inequalities. However, the wide spread use and acceptance of LP had to wait until the invention of the Standard Simplex Method.

1.1 Standard Simplex Method

Since World War II, LP has been used to solve problems of various dimensions in almost all disciplines. The most popular solution algorithm is the Simplex method that is implemented by most LP software packages.

The Simplex algorithm can be considered as a sub-gradient directional method, jumping from an initial feasible vertex to a neighboring vertex of the feasible region until it arrives at an optimal vertex. The Simplex method requires that the model be expressed in a special format called “Standard Form”, that is, some constraints and variables must be transformed. Consequently, when the manager obtains a solution through the Simplex method, she/he is left with the task of interpreting and transforming the Simplex solution back to the original managerial problem. However, this may not be an easy task.

For instance, solving LP problems in which some constraints are in (\geq) or ($=$) form with non-negative right-hand side (RHS) has raised difficulties. With that purpose, the simplex method requires a feasible starting solution. When such starting solution is not readily at-hand, alternative methods are necessary. One version of the Simplex method, known as the two-phase method, introduces an artificial objective function, which is the sum of artificial variables (see Arsham 1997a, 1997b). Another version adds penalty terms, which are the sum of artificial variables with very large, positive coefficients. The latter

approach is known as the Big-M method, (see Arsham 2006, 2007). On the other hand, using the Dual Simplex method has its own difficulties. For example, when some coefficients in the objective function are not dual feasible, one must introduce an artificial constraint. Handling equality ($=$) constraints by the dual simplex method is tedious because of the introduction of two new variables for each equality constraint: one extraneous slack variable and one surplus variable. Also, one may not be able to remove some equality ($=$) constraints by elimination at the outset, as this may violate the non-negativity condition introduced when constructing the “Standard Form.”

Arsham (2013) proposes a tabular interior boundary approach that intends to address these concerns and solves the original managerial LP model, without any need of “Standard Form”, artificial variables, artificial constraints, and Big-M. While these variants of the simplex successfully handle an array of constraint forms, they impose a burden and mathematical sophistication on manager that make difficult the success of LP applications.

1.2 The Costly Difference Between the Managerial and Modeler Interpretation

Koltai and Terlaky (2000) state that managerial questions are not answered satisfactorily with the mathematical interpretation of sensitivity analysis. Software packages provide sensitivity results focused on the optimality of a basis and not on the optimality of the values of the decision variables. The implementation of the misunderstood shadow prices and their range of validity may coax managers to take poor decisions with considerable financial losses and strategic consequences. Another source of confusion is the similarity of terms used in operations research (i.e., re-search, as a process), and other related fields. For example, “shadow price” and “opportunity cost” have somewhat different meanings in the LP

and Economics literature. The “opportunity cost” of an action in economics can be interpreted as the “shadow price” of that action on the budget.

The LP software packages provide sensitivity results about the optimality of a basis and not about the optimality of the values of the decision variables and the shadow prices that are of interest to the manager. In Section 6 we develop the largest sensitivity region based on the optimal vertex, and allows for simultaneous dependent/independent changes.

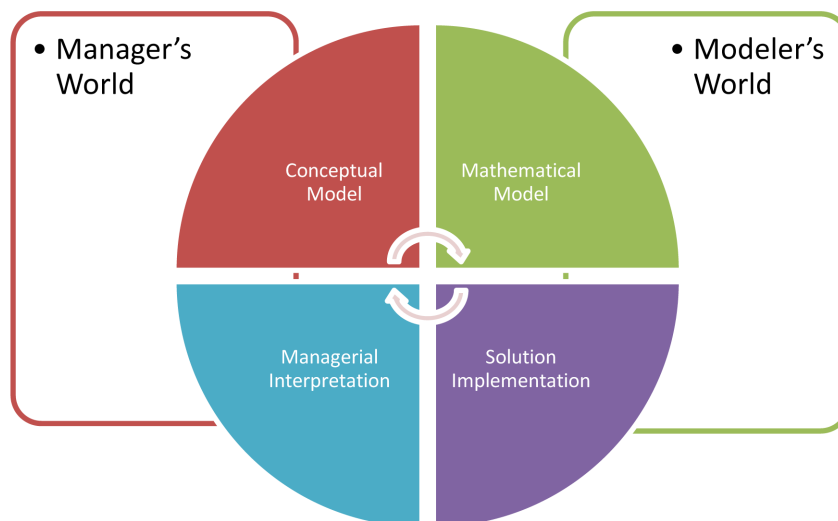
1.3 The Confusion Between the Model and the Solution Algorithm

Naturally, a modeler, reflecting the modeler’s decision problem, creates the LP model. On the other hand, the solution algorithm is designed by, for example, a systems analyst to solve LP problems. Because of the complexity of Simplex method, confusion could creep-in when the modeler’s model is developed into a tool using the software solution algorithm. For instance, the algorithm used in QM software and Management Scientists make all variables non-negative. This may create confusion as many LPs may contain negative or

unrestricted variables. In this research work, the algorithm adjusts itself to the original model and solves it, instead of changing the model. The decision maker easily understands the result, because our method does not change the structure of the original model. For example, while using Simplex approach the need to transform constraints to standard form changes the manager’s LP model to something that is not his and therefore the optimal solution must be transferred back to the manager’s problem. This can confuse the managers and lead to costly misinterpretations. One must understand the differences that could creep-in when a model is formulated and when it is solved using a solver’s solution algorithm; unfortunately the current state of art of LP software do not differentiate and causes confusions.

There exists a dichotomy between the reality the managers have to deal with, the problems that they face and the type of strategic decisions they need to make, and the methods that modelers have access to solve those problems. Research and technology have brought great advances to the decision-making process. However, more advances are needed to address directly the managers’ decision problems.

Figure 1. Manager’s world vs. modeler’s world



The main issue is that most software packages make assumptions about the structure of the problem, constraints have a certain direction; the decision variables take certain sign-constraints. These assumptions are without loss of generality; however, they do have implications that modify the original model and complicate the interpretation of the solution for implementation. As a result, the conceptual models from the manager's environment have to be translated into mathematical models, gets translated and later into solution models to fit software packages. Afterwards, these solutions must be translated back into the real problem. This translation is often cumbersome for the general user. While translating decision variable might be doable with some effort, interpreting sensitivity analyses from a modified model might be out of reach. In that regard the mathematical model should represent to the conceptual model.

We devoted Sections to show how to build the largest sensitivity region based on the "optimal solution". These sections are independent of all previous sections; in that one needs the optimal solution in any way it is obtained. This development differs significantly from the traditional Simplex's tableau approach that limits itself to the "Preservation of the Basis". No one expects the Manager to understand and interpret the Basis correctly, but it is expected that a Manager should be able to understand and interpret the optimal values of his decision variables and the consequences of any changes to them. The Dual generated from the "Standard Form" does not look like the Manager's dual problem of his primal model.

The Operations Research (OR) processes must be considered from the viewpoint of General Systems Theory. The authors assert that without a whole systems perspective OR can neither be understood nor applied effectively. Typically, OR problems can be broken down into five components viz: (1) the "reality" of the problem, (2) the conceptual model of the problem, (3) the conversion of conceptual model to a mathematical model, (4) solution determination of the mathematical

model, (5) managerial interpretation, and (6) the implementation of the solution. The authors believe that in most cases we have under-utilized our knowledge (study) and limited the application of the OR process. There have been extremely few studies and applications of OR that satisfies the systems point of view, Arsham (2012).

The remainder of this chapter is organized in thirteen sections. Almost every section can stand by itself, independent of previous sections. Section 2 introduces linear programming models from model builders and decision makers. Some oddities are included. Section 3 describes the decision maker's environment explicitly in order to enable them to understand the decision problem. Section 4 explores the feasible region by using algebra that is understandable by the decision maker. The algebraic approach to solve linear programs is given in Section 5, which includes computation of slack and surplus variables. The methodology solves LP model in an "as is" state, without any need of "Standard form", artificial variables, artificial constraints, the Big-M, and slack/surplus. Section 6 constructs the largest sensitivity region for the right hand side of the constraints based on the optimal solution. The shadow prices are byproducts of the sensitivity analysis. The dual problem formulation is included in Section 7, which is general and easy to apply. Section 8 considers the general sensitivity region for the cost coefficients of the objective function.

Throughout this chapter we have used LPs of two dimensions to save space but this method can be extended to higher dimensions. Even for two dimension problems we do not use the graphical method because it is limited to two-dimensional problems. Another reason to avoid graphical methods is that in some textbooks the graphical methods for finding the cost sensitivity range, and interpretation of the shadow prices are misleading. We use the algebraic approach throughout this chapter to overcome these limitations. The solution to a non-standard form problem is outlined in Section 9. Software implementation and their

limitations are examined in Section 10. The sensitivity analysis of the degenerate optimal vertex to remain optimal is developed in Section 11. Section 12 does sensitivity analysis of multiple optimal solutions. The existence of a curious property; known as More-For-Less, or Less-For-More is covered in Section 13. Concluding remarks are summarized in Section 14.

2. LINEAR PROGRAMMING MODEL

In general terms, a linear programming model is a mathematical approach for determining the set of decisions necessary to achieve the best outcome given a set of resource limitations and requirements represented as a system of linear equations. In business, these can be the outsourcing conditions and quantities to attain the lowest possible total cost, or the production plan to get the maximum profit, or the marketing mix to reach the highest market penetration.

2.1 Linear Programming Model

We define the LP model in the most general way.

Problem P:

Max (or Min) CX

Subject to $AX \leq a$,

$BX \geq b$,

$DX = d$,

$X_i \geq 0, i = 1, \dots, j$

$X_i \leq 0, i = j+1, \dots, k$

X_i unrestricted in sign, $i = k+1, \dots, n$

where matrices A , B , and D have p , q , and r rows, respectively with n columns and vectors c , a , b , and d have appropriate dimensions. Therefore, there are $m = (p + q + r + k)$ constraints and n decision variables. It is assumed that $m \geq n$. Note that the main constraints have been separated into three subgroups. Without loss of generality, we assume that all RHS elements, a , b , and d are non-negative.

We do not deal with trivial cases, such as where $A = B = D = 0$ (no constraints), or $a = b = d = 0$ (all boundaries pass through the origin point), or having LP with a single point feasible region. We assume the LP has a no redundant constraints.

2.2 What Can Go Wrong in the Process of Building a Linear Programming (LP) Model?

At the modeling stage potential pitfalls exist which affect any LP application; therefore, the decision-maker and the modeler should be cognizant of modeling phase of LP, Arsham (2008). Thus, potential problems exist which affect any linear programming application. Optimal solution may be infeasible or unbounded, or there may be multiple solutions. Degeneracy may also occur. Figure 2 presents a classification of LP for modeling validation process:

2.3 Dual Problem Construction

Associated with each (primal) LP problem there is a companion problem called its dual problem. Table 1 is a classification of the decision variable constraints is useful and easy to remember in construction of the dual.

General Principles

- If the primal is a maximization problem, then its dual is a minimization problem (and vice versa).
- For each constraint there is a dual decision variable (and vice versa).
- Use the variable type of one problem to find the constraint type of the other problem.
- Use the constraint type of one problem to find the variable type of the other problem.
- The RHS elements of one problem become the objective function coefficients of the other problem (and vice versa).

Figure 2. A classification of LP for modeling validation and sensitivity analysis

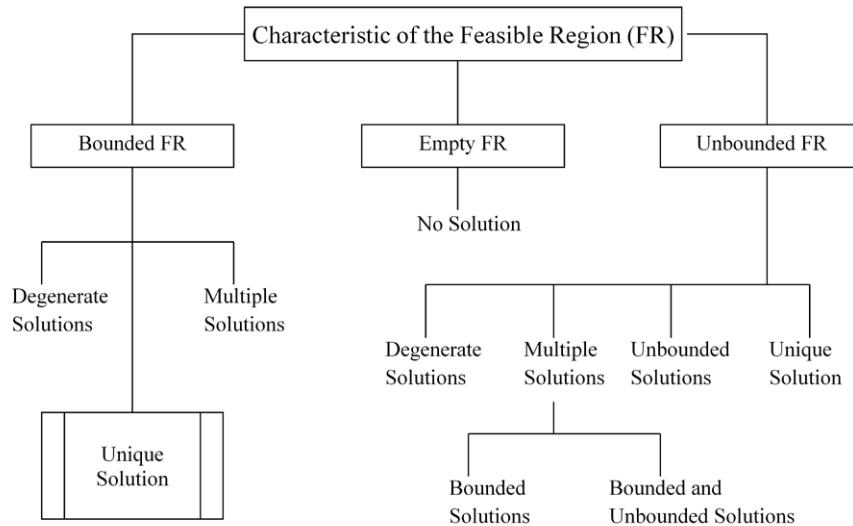


Table 1. A One-to-one correspondence between the primal and the dual LP problems.

The Dual Problem Construction		
Objective: Max (e.g. Profit)		Objective: Min (e.g. Cost)
Constraint types:		Constraint types:
\leq a Sensible constraint		\geq a Sensible constraint
$=$ a Restricted constraint		$=$ a Restricted constraint.
\geq an Unusual constraint		\leq an Unusual constraint.
	Variables types:	
	≥ 0 a Sensible condition	
	unrestricted in sign	
	≤ 0 an Unusual condition	

- The matrix coefficients of the constraints of one problem are the transpose of the matrix coefficients of the constraints for the other problem. That is, rows of the matrix become columns and vice versa.
- It may be more efficient to solve the dual LP problem than the primal Problem.
- The dual solution provides important economical interpretation such as shadow prices.
- If a constraint in the primal is non-binding, i.e., the LHS value is not equal to the RHS value, then the associated variable in the other problem is zero.

Applications

One may use duality in a wide variety of applications including:

Table 2. Possible combinations of primal and dual properties

Primal Problem	Condition Implies	Dual Problem
Feasible; bounded objective	↔	Feasible; bounded objective
Feasible; unbounded objective	→	Infeasible
Infeasible	←	Feasible; unbounded objective
Multiple solutions	↔	Degenerate solution
Degenerate solution	↔	Multiple solutions

- If a decision variable in the primal problem is not zero, then the associated constraint in the dual problem is binding.
- To obtain the sensitivity range of the RHS of primal problem from the sensitivity range of the cost coefficient in the other problem (and vice versa).

Table 2 implies some possible combinations of primal and dual problems.

Problems with LP Packages

Most LP software solvers have difficulties in recognizing the last two cases in the above table namely multiple optimal solutions and degenerate optimal solution. Most LP software do not alert the user about them yet produce sensitivity ranges that are not valid. One must make ensure that the solution is unique, and non-degenerate before analyzing and applying the sensitivity ranges.

2.4 Ensuring the Formulation Is Correct: Odd LP Models

Developing LP models with oddities can assist in the process of ensuring that the formulation is correct. However, almost all LP software solvers have difficulties in recognizing these dark sides of LP and/or giving any suggestions as remedies.

2.4.1 Unbounded Solutions

Identification

If the shadow prices of the primal problem do not satisfy the dual constraint, then the primal problem is unbounded.

Occurrence

An unbound optimal solution means the constraints do not limit the optimal solution and the feasible region is also unbounded, it effectively extends to infinity.

Resolution

In real life, this is not possible. Check the objective function. Possibly, the problem is a minimization problem and maximization was selected instead. Check the formulation of the constraints. Possibly, one or more constraints are missing. Check also the constraints for any incorrect specification in the direction of the inequality, and numerical errors.

2.4.2 Infeasibility

Identification

There is no feasible solution, i.e., there is no feasible vertex. The solution algorithm produces no solution.

Occurrence

An infeasible solution means the constraints are too limiting and have left no feasible region. That is, no solution satisfies all the constraints of the problem.

Resolution

Check the constraints for any incorrect specification in the direction of inequality constraints, and numerical errors. If no error exists, then there are conflicts of interests.

These and other odd situations, including multiple solutions and in particular degeneracy, are rare (except in network models since they often have a redundant constraint, causing degeneracy), and often result at the modeling stage. The decision-maker sets the objective function, while the constraints come from the decision-maker environment, consequently, it is rare to get multiple solutions, i.e. the objective function has the coefficient proportional to those of at least one binding constraint. In these cases, rounding in the coefficient(s) could be the source of the multiple solutions.

3. DECISION-MAKER'S ENVIRONMENT

In the process of conceptualizing and building a mathematical model of the business reality, it is important to identify the most important elements and categorize them into performance measures, uncontrollable inputs, parameters, controllable inputs and strategies.

Understanding the problem requires criteria for grouping entities of the decision model in the same category. Performance measure (or indicators), that is, measuring business performance, should be at the top of the decision-making process and should align with the management goals. The development of effective performance measures is increasingly important for many organizations,

even though it remains challenging for organizations in the public sector or not-for-profit organizations, because the performance measures are the drivers for the optimal strategy and determine the direction for the organization.

The decision-making environment elements are further discussed in the context of the following numerical example.

An Illustrative Numerical Example

Consider the following mixed products LP problem of a price-taker small manufacturer producing table and chairs. His objective is to maximize net profit:

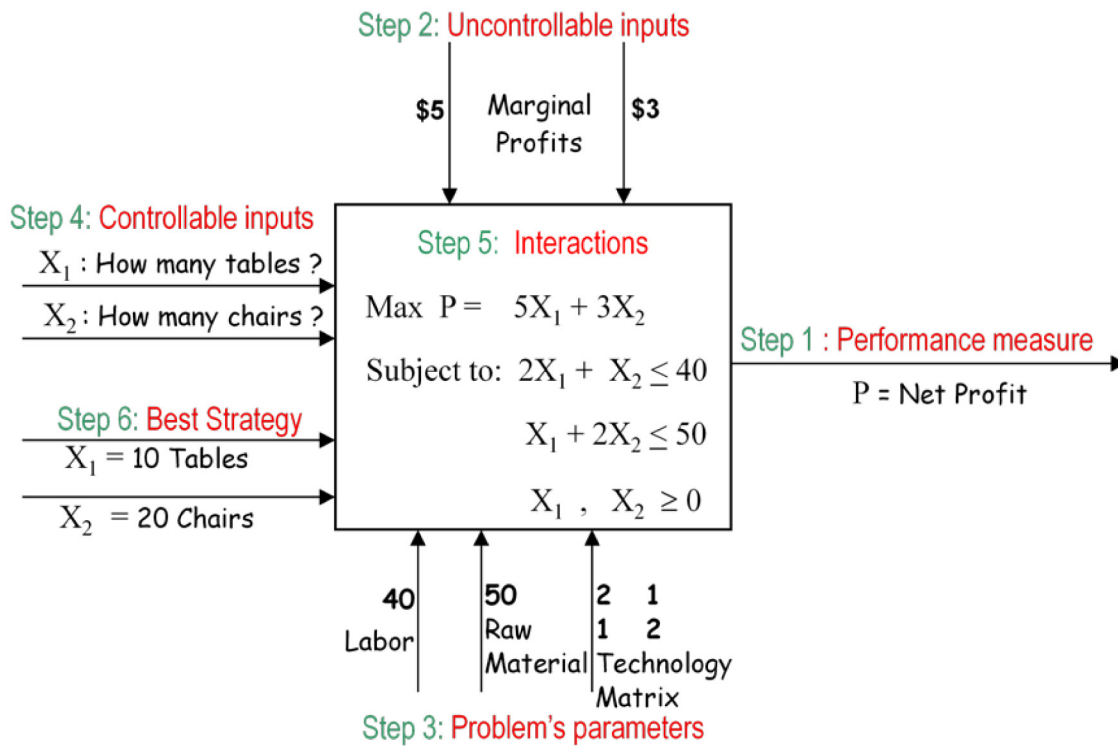
$$\begin{aligned} \text{Maximize } P(\mathbf{X}) &= 5X_1 + 3X_2 \\ \text{Subject to } 2X_1 + X_2 &\leq 40, \\ X_1 + 2X_2 &\leq 50, \\ X_1 &\geq 0, X_2 \geq 0. \end{aligned}$$

Figure 3 analyzes and depicts his decision environment.

The performance measure is important for the decision maker as a measuring tool for success and identifying the decision problem. Other elements of the decision-maker's environment are classified as follows:

- **Uncontrollable Inputs:** These come from the decision maker's external environment. Uncontrollable inputs often create the problem and constrain the actions.
- **Parameters:** Parameters are the constant elements that do not change during the time horizon of the decision review. These are the factors partially defining the problem. Strategic decisions usually have longer time horizons than both the tactical and the operational decisions, at different levels of organization.
- **Controllable Inputs:** The collection of all possible courses of action the decision maker might take.

Figure 3. Decision-maker's environment in a mixed product LP problem



- **Interactions among these Components:** These are logical, mathematical functions representing the cause-and-effect relationships among inputs, parameters, and outcomes.

problems, adding new uncontrollable inputs, and /or new options. One must identify and anticipate these new problems and recognize the cycle of such problems.

There are some constraints that are made up of all the parameters. Therefore, they do not need to be treated separately.

- **Actions:** Action is the ultimate decision and is the best course of strategy to achieve the desirable outcome. Decision-making involves the selection of a course of action in the pursuit of one's objective.
- **Controlling the Problem:** Few problems in life, once solved, stay that way. Changing conditions tend to create new problems from the problems that were previously solved. Their solutions create new

4. SOLVING THE SYSTEM OF LINEAR INEQUALITIES

The widely used Simplex solution to an LP problem is strictly based on the theory and solution of a system of linear inequalities (Arsham 2009). The list of constraints in an LP model constitutes a system of linear inequalities that define the feasible region of the problem. The basic solutions to a linear program are the solutions to the systems of equations consisting of constraints at binding positions. Not all basic solutions satisfy all the constraints. Only those that do meet all the constraint restrictions are called the basic feasible

solutions. The basic feasible solutions correspond precisely to the vertices of the feasible region.

For illustration purposes we are going to follow the Improved Algebraic Method (IAM) to solve the system of linear inequalities. This method is simple to understand and does not require the formulation of an auxiliary LP problem like it might require the Simplex method. This methodology can also be extended to solve systems of linear inequalities of higher dimensions.

We are interested in finding the vertices of the feasible region of Problem P, expressed as a system of linear equalities and inequalities.

Feasible Region S:

$$\begin{aligned} AX &\leq a, \\ BX &\geq b, \\ DX &= d, \end{aligned}$$

where some $X_i \geq 0, i = 1, \dots, j, X_i \leq 0, i = j+1, \dots, k$ and X_i unrestricted in sign, $i = k+1, \dots, n$. Matrices A, B, and D as well as vectors a, b, and d have appropriate dimensions.

Therefore, the optimization problem can be expressed as:

Problem P:
Max (or min) $C(\mathbf{X})$
Subject to: $\mathbf{X} \in S$

4.1 Steps of the Improved Algebraic Method (IAM)

- Step 1.** Convert all inequalities into equalities including any variable restricted in sign.
- Step 2.** Calculate the difference between the number of variables (n) and the number of equations (m).
- Step 3.** Determine the solution to all square systems of equations, i.e., basic solutions. The maximum number of systems of equations to be solved is: $m/[n!(m-n)!]$.

Step 4. Check feasibility of each basic solution obtained in Step 3 by using all the constraints and identify the basic feasible solutions (BFS).

The coordinates of vertices are the BFSs of the systems of equations obtained by setting some of the constraints at binding (i.e., equality) position. For a bounded feasible region, the number of vertices is at most $m/[n!(m-n)!]$ where m is the number of constraints and n is the number of variables. Therefore, a BS is obtained by taking any set of n equations then solving them, simultaneously. By plugging this BS in the constraints of other equations, one can check for feasibility of the BS. If it is feasible, then this solution is a BFS that provides the coordinates of a corner point of the feasible region.

4.2 Numerical Example For Solving System of Inequalities

We provide a feasible region problem; FR1 to illustrate the Improved Algebraic Method steps:

Feasible Region FR1:

$$\begin{aligned} 12X_1 + 10X_2 &\geq 12000 && (1) \\ 10X_1 + 15X_2 &\leq 15000 && (2) \\ X_1 &\geq 0 && (3) \\ X_2 &\geq 0 && (4) \end{aligned}$$

Because of constraints 2, 3 and 4, are enclosing feasible region, therefore feasible region is bounded with countable vertices. There will be 4 equations and 2 variables yielding 6 possible combinations. The IAM provides six basic solutions for the system of linear inequalities as shown in Table 3.

There are 3 basic feasible solutions to the system of inequalities namely:

$$\begin{aligned} A &= (X_1, X_2) = (1000, 0), \\ B &= (X_1, X_2) = (1500, 0), \\ C &= (X_1, X_2) = (375, 750). \end{aligned}$$

Table 3. The basic solutions of the numerical example FR1

Binding Constraints	Solution (X_1, X_2)	Feasible? Why?
1 and 2	(375, 750)	Yes, all satisfied
1 and 3	(0, 1200)	No, constraint 2 is not satisfied
1 and 4	(1000, 0)	Yes, all satisfied
2 and 3	(0, 1000)	No, constraint 1 is not satisfied
2 and 4	(1500, 0)	Yes, all satisfied
3 and 4	(0, 0)	No, constraint 1 is not satisfied

4.3 Algebraic Presentation of Feasible Region

Using the scalar parameters $\lambda_1, \lambda_2,$ and λ_3 for the 3 vertices, respectively, we obtain the following convex combination of vertices which is a parametric representation of the bounded feasible region:

$$X_1 = 1000\lambda_1 + 1500\lambda_2 + 375\lambda_3$$

$$X_2 = 750\lambda_3$$

for all parameters $\lambda_1, \lambda_2, \lambda_3$ such that each is non-negative, and $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Note that the above representation is valid if the feasible region is bounded. For example the feasible region $X_1 + X_2 \leq 1, X_1 \geq 1, X_2 \geq 0,$ is unbounded contains one vertex only.

Moreover if feasible region is bounded it must have more than one vertex. The feasible region of $X_1 + X_2 = 1, X_1 - X_2 = 1, X_1 \geq 0, X_2 \geq 0,$ has feasible region with single point (1, 0). In the case of unbounded feasible region one must includes the needed rays, for numerical examples, see Arsham (2008).

Some Useful Comment on Algebraic Solution Method (IAM)

The main purpose of presenting the IAM is for the manager to understand, use, gets useful information, the major software use a refine version of IAM. Therefore, for large problems based on

software become apparent. There are special cases when using IAM:

- If there is no feasible vertex, then the problem is infeasible, that is feasible region is empty.
- If the shadow price of the primal does not satisfy the constraints of the dual problem then the problem is unbounded.
- If there are two distinct optimal vertices, then the problem has multiple solutions. In this case the dual has degenerate optimal solution. The reverse is also true.

Notice that these results are not applicable to trivial cases, such as a problem with no vertices, such as: Maximize $X_1 + X_2,$ subject to: $X_1 + X_2 = 1,$ all variable are unrestricted in sign. For more oddities in LP modeling, visit the website: The Tools for Modeling Validation Process: The Dark Side of LP <http://home.ubalt.edu/ntsbarsh/opre640A/partv.htm>

5. SOLVING LINEAR PROGRAMS

Consider the following linear program P1:

Problem P1:

$$\begin{aligned} \text{Minimize } C(\mathbf{X}) &= 18X_1 + 10X_2 \\ \text{Subject to: } &12X_1 + 10X_2 \geq 12000 \\ &10X_1 + 15X_2 \leq 15000 \\ &X_1, X_2 \geq 0 \end{aligned}$$

Substituting the parametric version of the feasible region into the objective function, we obtain:

$$C(\lambda) = 18X_1 + 10X_2 = 18000\lambda_1 + 27000\lambda_2 + 14250\lambda_3$$

The optimal solution occurs when $\lambda_3 = 1$ and all other λ_i 's are set to 0, with a minimum value of 14250. The optimal solution is ($X_1 = 375$, $X_2 = 750$), the vertex corresponding to λ_3 .

Let the terms with the largest (smallest) coefficients in $C(\lambda)$ be denoted by λ_L and λ_S respectively. Then the following proposition holds:

Proposition 1: *The maximum (minimum) solution of an LP with a bounded feasible region correspond to the maximization (minimization) of the parametric objective function $C(\lambda)$. Moreover, since $C(\lambda)$ is a linear convex combination, the optimal value of $C(\lambda)$ is obtained by setting λ_L or λ_S equal to 1 and all other $\lambda_i = 0$.*

Proof: Proof follows from applying the IAM to

$$\begin{aligned} &\text{Max (Min) } C(\lambda) \\ &\text{Subject to: } \sum \lambda_i = 1, \text{ and all } \lambda_i \geq 0. \end{aligned}$$

The optimal solution can be found by enumerating all basic feasible solutions by applying the IAM method to this optimization problem. Since this feasible region is bounded it has a finite number of vertices, then this result suggests that optimum is associated with the BFS yielding the largest or smallest objective value, assuming the problem is of maximization or minimization type, respectively. The maximum and minimum vertex of an LP correspond to $\lambda_L = 1$ and $\lambda_S = 1$, respectively. As a result, since $C(\lambda)$ is a (linear) convex combination of its coefficients, the optimal solution of $C(\lambda)$ is obtained by setting λ_L or λ_S equal to 1 and all other $\lambda_i = 0$.

As an application of this proposition, we have the following useful result.

Fundamental Linear Programming Result: *if the optimal solution to an LP is bounded, then it occurs at one of vertices of the feasible region.*

From the above proposition, one might find the tight bound for the objective function, if it exists. For our numerical example we have the following tight bound:

$$14250 \leq 18X_1 + 10X_2 \leq 27000$$

5.1 Computing Slack and Surplus

Given that the right-hand-side (RHS) of a constraint is non-negative, the slack is the leftover amount of a resource (\leq) constraint, and surplus is the excess over a requirement (\geq) constraint. These quantities represent the absolute values of the difference between the RHS value and the LHS (Left Hand Side) evaluated at an optimal point. Having obtained an optimal solution, one can compute the slack and surplus for each constraint at optimality. Equality constraints are always binding with zero slack/surplus.

Since this numerical example is a two-dimensional LP, one expects (at least) to have two binding constraints. The binding constraints at optimality are equations 1 and 2:

$$\begin{aligned} 12X_1 + 10X_2 &= 12000 \\ 10X_1 + 15X_2 &= 15000 \end{aligned}$$

With surplus and slack value of zero, respectively, i.e., $S_1 = 0$ and $S_2 = 0$ which are the “Reduce Cost” of the dual problem.

6. CONSTRUCTION OF THE LARGEST SENSITIVITY REGIONS BASED ON OPTIMAL SOLUTION NOT “BASIS”

Suppose we obtained the optimal solution by any means, such as graphical (for two dimen-

sional problems), simplex, dual simplex or interior methods. The following two sections construct the largest sensitivity regions based on the given optimal solution.

We develop a novel approach to post-optimality analysis for general LP problems given a unique non-degenerate optimal solution this approach provides a simple framework for the analysis of any single or simultaneous change of right hand side (RHS) or cost coefficients by solving the nominal LP problem with perturbed RHS terms. Assume that the LP is of size (mxn), i.e., it has n decision variables and m constraints including the non-negativity conditions (if any).

6.1 Computation of Shadow Prices

- Step 1:** Identify the n constraints that are binding at optimal solution. If there is more than n binding constraints, then the problem is degenerate. The degenerate case are dealt with in Section 10.
- Step 2:** Construct the parametric RHS of the constraints, excluding the non-negativity conditions.
- Step 3:** Solve the parametric system of equations consisting of the binding constraints. This provides the parametric optimal solution.
- Step 4:** Construct the simultaneous sensitivity analysis by plugging-in the solution obtained in Step 3, in all other remaining parametric constraints, including the non-negativity conditions (if any).
- Step 5:** Plug in the parametric optimal solution into the objective function. The coefficient for each parameter is the shadow price for that binding constraint. By definition, the shadow price for a non-binding constraint is always zero.

Following the above steps for our numerical example, one must first solve the following RHS parametric system of equations. By setting all inequalities in binding positions and solving the

system of equations, we get the following parametric solution:

$$\begin{aligned} 12X_1 + 10X_2 &= 12000 + r_1 \\ 10X_1 + 15X_2 &= 15000 + r_2 \end{aligned}$$

Solving this parametric system of equation for X1 and X2, we have:

$$\begin{aligned} X_1 &= 375 + 3/16r_1 - 1/8r_2 \\ X_2 &= 750 - 1/8r_1 + 3/20r_2 \end{aligned}$$

The solution can be verified by substitution. For the higher dimension LP, the parametric solution can be obtained by using the JavaScript:

<http://www.mirror-service.org/sites/home.ub-alt.edu/ntsbarsh/Business-stat/otherapplets/PaRHSSyEqu.htm>

For large-scale problems one may use symbolic software such as Maple, Matlab, or Mathematica.

The parametric optimal solution shows the impact of changes on the RHS on the optimal decision variables. For example, if the RHS of constraint 1 increases the quantity of X₁ will increase and X₂ decreases. This is often a concept misunderstood or overlooked by managers.

Plugging the parametric solution into objective function, we have:

$$18X_1 + 10X_2 = 14250 + 17/8r_1 - 3/4r_2$$

The shadow prices are the coefficients of the parametric optimal function, i.e., U₁ = 17/8 and U₂ = -3/4, for the RHS of constraints 1 and 2, respectively. They are the *rate of change* in optimal value with respect to changes in the RHS of each constraint.

6.3 Construction of the RHS Sensitivity Region: Maintaining the Validity of Current Shadow Prices

Notice that the parametric objective function $14250 + 17/8r_1 - 3/4r_2$ is valid when the parametric solution satisfies all other (i.e., non-binding) constraints. In the current numerical example, the other constraints are:

$$X_1 \geq 0 \text{ and } X_2 \geq 0.$$

These produce the following conditions:

$$375 + 3/16r_1 - 1/8r_2 \geq 0, \text{ and } 750 - 1/8r_1 + 3/20r_2 \geq 0.$$

Notice that this is the largest sensitivity region for the RHS of the binding constraints that allows for simultaneous, dependent/independent changes. This convex region is non-empty since it always contains the origin ($r_1 = 0, r_2 = 0$), with a vertex at ($r_1 = -12000, r_2 = -15000$) where both right-hand sides binding constraints.

Arsham (1990) gives detail treatments of the most popular sensitivity analysis, including the 100% rule, and the tolerance analysis.

6.4 Sensitivity Range for Each RHS of the Binding Constraints

The above set of inequality sensitivity region can be used to find the ordinary sensitivity range (one change-at a time) for the RHS values of the constraints. The range for the RHS of the first constraint (RHS_1) can be obtained by setting $r_2 = 0$ in the inequalities set (2). This provides $r_1 \leq 6000$ and $r_1 \geq -2000$. Therefore, the allowable increase and decrease in the original value 12000 for RHS_1 are 6000 and 2000, respectively, i.e., the current set of shadow prices remain valid as long as:

$$10000 \leq RHS_1 \leq 18000 \text{ and } RHS_2 = 15000.$$

Similarly, the range for the RHS of the second constraint (RHS_2) can be obtained by setting $r_1 = 0$ in the inequality set (2). This implies that $r_2 \leq 3000$ and $r_2 \geq -5000$. Therefore the allowable increase and decrease in the original value 15000 for RHS_2 are 3000 and 5000, respectively, i.e., the current set of shadow prices remain valid as long as:

$$RHS_1 = 12000 \text{ and } 10000 \leq RHS_2 \leq 18000.$$

6.5 Sensitivity Range for the Non-Binding Constraints

In the numerical example P1, the non-binding constraints are the non-negativity conditions which are not subject to sensitivity analysis. Consequently, for illustration purposes let us pretend “as if” there is a non-binding constraint $X_1 + 100X_2 \geq 20000$. To find the sensitivity range (one-change-at-a-time), construct the parametric form and plug in the optimal solution. We get

$$\begin{aligned} X_1 + 100X_2 &\geq 20000 + r_3 \\ 375 + 100(750) &= 75375 \geq 20000 + r_3 \\ r_3 &\leq 55375 \end{aligned}$$

The amount of increase is 55375 and decreasing amount is unlimited; i.e., ($RHS_3 \leq 75375$).

The following proposition formalizes the shifting of parametric non-binding constraints.

Proposition 2: For any given point $X^o = (X_1^o, X_2^o, \dots, X_n^o)$ the parameter r value for any resource/production constraint is proportional to the ordinary distance between the point X^o and the hyper-plane of the constraint.

Proof: Proof follows from the fact that the distance from point $X^o = (X_1^o, X_2^o, \dots, X_n^o)$ to any non-binding constraint, i.e.

$$a_1 X_1^o + a_2 X_2^o + \dots + a_n X_n^o = b + r$$

is

$$\text{Absolute } [a_1 X_1^o + a_2 X_2^o + \dots + a_n X_n^o - b - r] / (a_1^2 + a_2^2 + \dots + a_n^2)^{1/2}$$

This reduces to:

$$\text{Absolute } r / (a_1^2 + a_2^2 + \dots + a_n^2)^{1/2}.$$

Therefore the parameter r value is proportional to the distance with a constant proportionality $1 / (a_1^2 + a_2^2 + \dots + a_n^2)^{1/2}$. This is independent of point X^o . In the above example, X^o is the optimal vertex. This completes the proof.

6.6. Behavior of Changes in the RHS Values of the Optimal Value

To study the directional changes in the optimal value with respect to changes in the RHS (with no redundant constraints present, and all RHS ≥ 0), we distinguish the following two cases:

Case I: Maximization problem

For \leq constraint: The change is in the same direction. That is, increasing the value of RHS does not decrease the optimal value. It increases or remains the same depending on whether the constraint is a binding or non-binding constraint.

For \geq constraint: The change is in the reverse direction. That is, increasing the value of RHS does not increase the optimal value. It decreases or remains the same depending on whether the constraint is a binding or non-binding constraint.

For = constraint: The change could be in either direction (see the More-for-less section).

Case II: Minimization problem

For \geq type constraint: The change is in the reverse direction. That is, increasing the value of RHS does not increase the optimal value (rather, it decreases or has no change

depending on whether the constraint is a binding or non-binding constraint).

For \leq type constraint: The change is in the same direction. That is, increasing the value of RHS does not decrease the optimal value (rather, increases or has no change depending on whether the constraint is a binding or non-binding constraint).

For = constraint: The change could be in either direction (see the More-for-less section).

7. GENERAL SENSITIVITY REGION FOR COST COEFFICIENTS: MAINTAINING THE VALIDITY OF CURRENT OPTIMAL VERTEX NOT “BASIS”

Knowing the unique optimal solution for the dual problem by using any LP solver, one may construct the simultaneous sensitivity analysis for all coefficients of the objective function of the primal problem as follows. Assume that the dual problem has n decision variables and m constraints, including the non-negativity conditions (if any).

7.1 Steps in Finding Sensitivity Region for the Objective Function Coefficients

Step 0: Construct and solve the dual problem.

Step 1: Identify the n constraints that are binding at optimal solution (that is, at the shadow prices of the primal problem). If there are more than n constraints binding, then the primal problem may have multiple solutions.

Step 2: Construct the parametric RHS of the constraints, excluding the non-negativity conditions.

Step 3: Solve the parametric system of equations consisting of the binding constraints. This provides the parametric optimal solution.

Step 4: Construct simultaneous sensitivity region by plugging in the parametric solu-

Table 4.

Primal Problem P1:	Classifications
Minimize $C(X) = 18X_1 + 10X_2$	Minimization Problem
Subject to: $12X_1 + 10X_2 \geq 12000$	Sensible Constraint
$10X_1 + 15X_2 \leq 15000$	Unusual Constraint
$X_1 \geq 0$	<i>Sensible Condition</i>
$X_2 \geq 0$	<i>Sensible Condition</i>
The Dual Problem of Problem P1 is:	
Maximize $12000U_1 + 15000U_2$	Maximization Problem
Subject to: $12U_1 + 10U_2 \leq 18$	<i>Sensible Constraint</i>
$10U_1 + 15U_2 \leq 10$	<i>Sensible Constraint</i>
$U_1 \geq 0$	Sensible Condition
$U_2 \leq 0$	Unusual Condition

tion obtained in Step 3, in all non-binding constraints, including the sign conditions such as non-negativity conditions (if any).

We demonstrate the method in our example. The dual problem of our numerical example is constructed in Table 4.

The components of the optimal solution to the dual problem are the shadow prices to the original (primal) LP model, and vice versa. The shadow prices of the primal, i.e., ($U_2 = 17/8$, $U_2 = -3/4$), as calculated earlier (or one can solve it afresh without referring to previous section). The optimal value for the dual $12000(17/8) + 15000(-3/4) = 14250$ that is equal to the optimal value of the primal problem, as always expected. The property that let the primal and the dual optimal values to be equal is known as strong duality, i.e., there is no gape between the dual and primal problem. The slacks of the first two constraints are $12(17/8) + 10(-3/4) - 18 = 0$, $10(17/8) + 15(-3/4) - 10 = 0$, which are the “Reduce Cost” of the primal problem.

To find the ranges for the objective function coefficients, one may use the RHS parametric version of the dual problem;

The parametric presentations of the RHS of the binding constraints are as follows:

$$12U_1 + 10U_2 = 18 + c_1$$

$$10U_1 + 15U_2 = 10 + c_2,$$

Solving these parametric equations we obtain

$$U_1 = 17/8 + 3/16c_1 - 1/8c_2$$

$$U_2 = -3/4 - 1/8c_1 + 3/20c_2$$

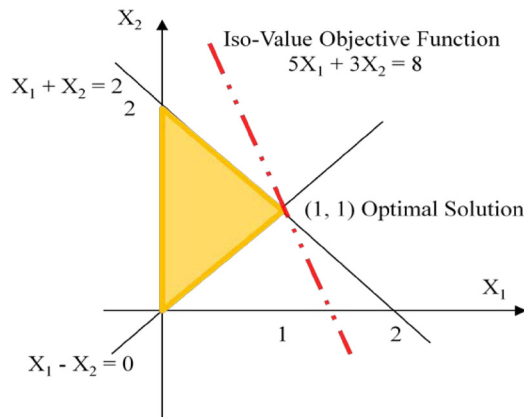
Plugging this solution into objective function of the dual problem, we obtain the parametric objective function is $14250 + 375c_1 + 750c_2$ with optimal value of 14250, same as for the optimal value of the primal problem, as expected. Again, this parametric optimal solution is subject to satisfying the unused constraints; namely, $U_1 \geq 0$ and $U_2 \leq 0$. These produce the following largest sensitivity region for the objective function coefficients of the primal problem P1, simultaneously:

$$3/16c_1 - 1/8c_2 \geq -17/8 \text{ and } -1/8c_1 + 3/20c_2 \leq 3/4.$$

Notice that this is the largest sensitivity region for cost coefficients, allowing for simultaneous, dependent/independent changes. This convex region is non-empty because it contains the origin ($c_1 = 0$, $c_2 = 0$), with a vertex at ($c_1 = -18$, $c_2 = -10$) which is the negative of the both cost coefficients.

The ordinary sensitivity range (one change at a time) of the coefficient of first decision variable X_1 , currently at 18, can be found by setting $c_2 = 0$ in the above sensitivity region inequality set, yielding $c_1 \geq -6$ and $c_1 \geq -34/3$. Therefore the allowable decrease for the coefficient of X_1 is 6 with unbounded (i.e., Big-M) allowable increase. Similarly, the sensitivity range of the coefficient of second decision variable X_2 , currently at 10, can be found by setting $c_1 = 0$ in the above sensitivity region inequality set, yielding $c_2 \leq 5$ and $c_2 \leq 17$. Hence, $c_2 \leq 5$, therefore, the allowable increase for the coefficient of X_1 is 5 with unbounded allowable decrease, (i.e., - Big-M).

Figure 4. Problem P3 feasible region



7.2 Finding the Cost Sensitivity Range by the Graphical Method Could Be Misleading

It is a commonly held belief that one can compute the cost sensitivity range by bracketing the slope of the (iso-value) objective function by the slopes of the two lines resulting from the binding constraints. This graphical slope-based method to compute the sensitivity ranges is described in popular textbooks, such as Anderson et al., (2007), Lawrence and Pasternack (2002), and Taylor (2010). However, this approach is only valid when the objective function and the two binding constraints have all slopes with the same sign, that is, all three functions have negative slopes or all three functions have positive slopes. The approach fails if one of the binding constraints has a non-negativity condition or the binding constraints have opposite signs. The following counterexample illustrating the later case.

Problem P3:

$$\begin{aligned} &\text{Maximize } 5X_1 + 3X_2 \\ &\text{Subject to: } X_1 + X_2 \leq 2 \\ &\quad X_1 - X_2 \leq 0 \\ &\quad X_1 \geq 0, X_2 \geq 0 \end{aligned}$$

The optimal solution, $(X_1, X_2) = (1, 1)$, is determined by the two main constraints. The iso-value objective function line and first constraint boundary line have negative slopes, while the other constraint has positive slope. Bracketing the objective function slope between the slopes of the two constraints gives $-1 \leq -C_1/C_2 \leq 1$, which is an incorrect range. Notice that, not even the current objective function slope $-5/3$ belongs to this range. The main reason is that in this problem to transition the slope from negative to positive it has to go through being very large and negative, then undefined, and then very large and positive, as opposed to go through zero. Then, the correct ranges are $C_1 \geq 3$, and $-5 \leq C_2 \leq 5$, respectively, found by equating the ratio of coefficients from the objective function plus and from each of the constraints as follow:

$$\begin{aligned} (5+c_1)/1 &= 3/1 \text{ gives } c_1 = -2 \\ (5+c_1)/1 &= 3/-1 \text{ gives } c_1 = -8 \end{aligned}$$

where c_1 represents the largest possible change in the coefficient. A positive number represents the largest possible increase, while a negative number represents the largest possible decrease.

Therefore, since $c_1 = -2$ is more restrictive than $c_1 = -8$, you may decrease $C_1 = 5$, by 2. There is no lower limit, thus $C_1 \geq 3$.

On the other hand, for C_2 we obtain an upper and lower bound

$$\begin{aligned} (3+c_2)/1 &= 5/1 \text{ gives } c_2 = 2 \\ (3+c_2)/-1 &= 5/1 \text{ gives } c_2 = -8 \end{aligned}$$

Therefore you may decrease $C_2 = 3$, by 8, $-5 \leq C_2$, and you may increase $C_2 = 3$, by 2, i.e. $C_2 \leq 5$.

7.3 Shadow Prices as the Lagrangian Multipliers

The dual solution provides important economical interpretation such as the marginal values of the RHS elements. The elements of the dual solution

are known as the Lagrangian multipliers because they provide (a tight) bound on the optimal value of the primal, and vice versa. For example, considering our numerical example the dual solution can be used to find a lower tight bound for the optimal value, as follow: Multiply each constraint by its corresponding dual solution and then add them up, we get:

$$\begin{array}{r} 17/8 [12X_1 + 10X_2 \geq 12000] \\ -3/4 [10X_1 + 15X_2 \leq 15000] \\ \hline 18X_1 + 10X_2 \geq 14250 \end{array}$$

Notice that the resultant on the left side is the objective function of the primal problem, and this lower bound is a tight one, since the optimal value is 14250. In other words, the objective function lower bound is 14250, which is the optimal value for the minimization of the objective function $18X_1 + 10X_2$. Moreover, the optimal value for both problems, dual and primal is always the same. This fact is referred to as equilibrium of economical systems, and efficiency in Pareto's sense between the Primal and the Dual Problems. Therefore, there is no duality gap in linear programming.

7.4 Misinterpretation of the Shadow Price

The Shadow Price tells us how much the objective function will change if we change the right-hand side of the corresponding constraint. This is often called the "marginal value", "dual prices" or "dual value", for the constraint. Therefore, the shadow price may not be same as the "Market price, i.e., real price".

For each RHS constraint, the Shadow Price tells us exactly how much the objective function will change if we change the RHS of the corresponding constraint within the limits given in the sensitivity range on the RHS's. Therefore, for each RHS value, the shadow price is the rate of

change in the optimal value caused by any allowable increase or decrease in the RHS.

Unfortunately, there are misconceptions regarding the definition of the shadow price. One such misinterpretation is, "In linear programming problems the shadow price of a constraint is the difference between the optimized value of the objective function and the value of the objective function, evaluated at the optional basis, when the right hand side (RHS) of a constraint is increased by one unit." "Shadow Prices: The shadow prices for a Linear Programming problem are the solutions to its dual. The i^{th} shadow price is the change in the objective function resulting from a one-unit increase in the i^{th} coordinate of b . A shadow price is also the amount that an investor would have to pay for one unit of a resource in order to buy out the manufacturer."

Consider the following LP as a counterexample:

Problem P3:

$$\begin{array}{l} \text{Maximize } X_2 \\ \text{Subject to: } X_1 + X_2 \leq 2 \\ \quad \quad \quad 2.5X_1 + 4X_2 \leq 10 \\ \quad \quad \quad X_1 \geq 0, X_2 \geq 0 \end{array}$$

This problem attains its optimal solution at (0, 2) with an optimal value of 2. Suppose we wish to compute the shadow price of the first resource that is the RHS of the first constraint. Changing the RHS of the first constraint by increasing it by one unit results in:

$$\begin{array}{l} \text{Maximize } X_2 \\ \text{Subject to: } X_1 + X_2 \leq 3 \\ \quad \quad \quad 2.5X_1 + 4X_2 \leq 10 \\ \quad \quad \quad X_1 \geq 0, X_2 \geq 0 \end{array}$$

The new problem has the optimal solution (0, 2.5) with an optimal value of 2.5.

Therefore, it seems "as-if" the shadow price for this resource is $2.5 - 2 = 0.5$. In fact the shadow

price for this resource is 1, which can be found by constructing and solving the dual problem.

The reason for this error becomes evident if we notice that the allowable increase to maintain the validity of the shadow price of the first resource is 0.5. The increase by 1 is beyond the allowable change on the first RHS value.

Now suppose we change the same RHS value by, increasing say, by 0.1, which is permissible (compared with its nominal value of 2), then the optimal value for the new problem is 2.1. Therefore the shadow price is $(2.1 - 2) / 0.1 = 1$. We must be careful when calculating shadow prices using this approach the change must be within allowable limits.

If you wish to compute the shadow price of a RHS when its sensitivity range is not available, you may obtain the optimal values for at least two perturbations. If the rate of change for both cases gives you the same values, then this rate is indeed the shadow price. As an example, suppose we perturb the RHS of the first constraint by +0.02 and -0.01. Resolving the problem after these changes using your LP solver, the optimal values are 2.02, and 1.09, respectively. Since the optimal value for the nominal problem (without any perturbation) is equal to 2, the rate of change for the two cases are: $(2.02 - 2)/0.02 = 1$, and $(1.09 - 2)/(-0.01) = 1$, respectively. Since these two rates are the same, we conclude that the shadow price for the RHS of the first constraint is indeed equal to 1.

7.5 Managerial Round-Off Errors

You must be careful whenever you round the value of shadow prices. For example, the shadow price of the resource constraint in the above problem is 8/3; therefore, if you wish to buy more of this resource, you should not pay additional price more than \$2.66. Whenever you want to sell any unit of this resource, you should not sell it at an additional price below \$2.67.

A similar error might occur whenever you round the limits on the sensitivity ranges. One

must be careful because the upper limit and lower limit must be rounded down and up, respectively.

8. DEALING WITH NON-STANDARD FORM LP PROBLEMS

Creating the Non-Negativity Conditions

By default, most LP software such as Management Scientist (1999), and QM for Windows (2003), assume that all variables are non-negative.

To achieve this requirement, convert any unrestricted variable X_j to two non-negative variables by substituting $y - X_j$ for every X_j . This increases the dimensionality of the problem by only one (introduce one y variable) regardless of how many variables are unrestricted.

If any X_j variable is restricted to be non-positive, substitute $-X_j$ for every X_j . This reduces the complexity of the problem.

Solve the converted problem and then substitute these changes back to get the values for the original variables and optimal value is straightforward. However, to convert sensitivity analyses including the shadow prices are not easy tasks. Consider the following LP problem:

Problem P4:

Maximize $X_1 + 0X_2$

Subject To:

$$X_1 + X_2 \geq 0$$

$$2X_1 + X_2 \leq 2$$

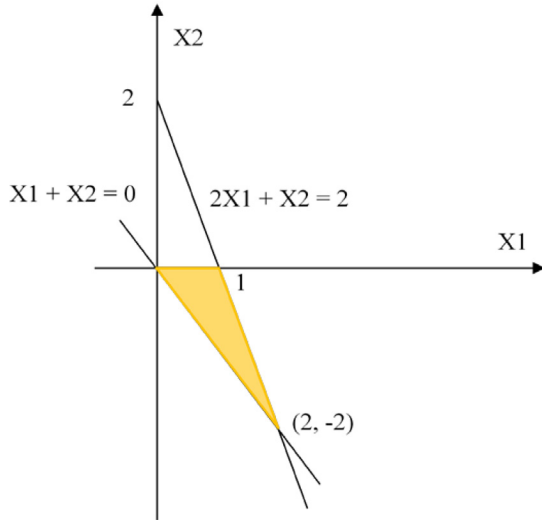
$$X_1 \geq 0$$

$$X_2 \leq 0$$

Following the IAM solution algorithm, optimal solution is at vertex $X_1 = 2, X_2 = -2$, which is depicted in Figure 5.

Performing the sensitivity analysis based on optimal vertex to maintaining the optimal vertex, we construct the following parametric RHS of binding constraints:

Figure 5. A typical feasible region for a non-standard form LP



$$\begin{aligned} X_1 + X_2 &= 0 + R_1 \\ 2X_1 + X_2 &= 2 + R_2 \end{aligned}$$

The parametric optimal solution is:

$$\begin{aligned} X_1 &= 2 - R_1 + R_2 \\ X_2 &= -2 + 2R_1 - R_2 \end{aligned}$$

We plug the parametric solution into objective function, we have:

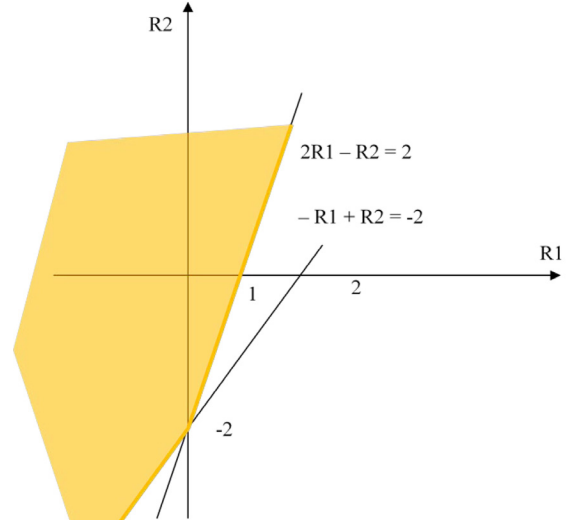
$$X_1 = 2 - R_1 + R_2$$

The shadow prices are the coefficients of the parametric optimal function, i.e., $U_1 = -1$ and $U_2 = 1$, for the RHS of constraints 1 and 2, respectively. They are the *rate of change* in optimal value with respect to changes in the RHS of each constraint.

The shadow prices remain valid as long as this parametric optimal solution satisfy all the other constraints, i.e., $X_1 \geq 0$ and $X_2 \leq 0$. This gives the largest sensitivity region for the RHS.

$$\begin{aligned} 2 - R_1 + R_2 &\geq 0 \\ -2 + 2R_1 - R_2 &\leq 0 \end{aligned}$$

Figure 6. The largest sensitivity region for the RHS of first and second constraints



The sensitivity region is convex and non-empty, containing the origin (0, 0) and its vertex is at (0, -2), as shown in Figure 4.

Figure 6 presents the largest sensitivity region for the RHS for which the shadow prices remain unchanged, i.e., the optimal dual problem vertex remains optimal.

The ordinary sensitivity range is $R_1 \leq 1$ for the first RHS and $R_2 \geq -2$ for the second, as can be verified from the largest sensitivity region for the RHS depicted in Figure 4.

We now perform sensitivity analysis on the objective function coefficient. The dual problem is constructed in Table 5.

The optimal solution is the shadow prices ($U_1 = -1$, $U_2 = 1$), with optimal value of $0U_1 + 2U_2 = 2$, same as primal, as expected.

$$\begin{aligned} U_1 + 2U_2 &= 1 + C_1 \\ U_1 + U_2 &= 0 + C_2 \end{aligned}$$

The dual parametric solution is:

$$\begin{aligned} U_1 &= -1 - C_1 + 2C_2 \\ U_2 &= 1 + C_1 - C_2 \end{aligned}$$

Table 5.

The Primal Problem: Maximize $X_1 + 0X_2$ Subject To: $X_1 + X_2 \geq 0$ $2X_1 + X_2 \leq 2$ $X_1 \geq 0$ $X_2 \leq 0$	Classifications Maximization Unusual Sensible Sensible Unusual
Therefore the dual formulation is:	
The Dual Problem: Minimize $0U_1 + 2U_2$ Subject To: $U_1 + 2U_2 \geq 1$ $U_1 + U_2 \leq 0$ $U_1 \leq 0$ $U_2 \geq 0$	Minimization Sensible Unusual Unusual Sensible

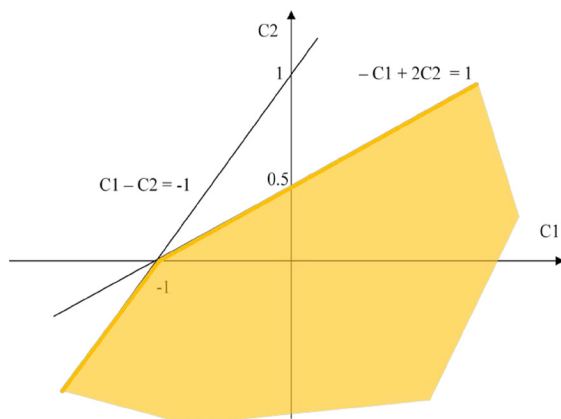
The solution must satisfy the other constraints:

$$-1 - C_1 + 2C_2 \leq 0$$

$$1 + C_1 - C_2 \geq 0$$

Figure 7 depicts the largest sensitivity region for the coefficients of objective function, i.e., the optimal solution remains unchanged, i.e., the

Figure 7. The largest sensitivity region for the coefficients of objective function



optimal vertex for the primal problem remains optimal.

One Change at a Time (i.e., Ordinary Sensitivity Range)

Set $C_2 = 0$, then $C_1 \geq -1$, that is, the allowable decrease is -1, and allowable increase is big-M (i.e., a large nonspecific positive number). Set $C_1 = 0$, Then $C_2 \leq 1/2$, that is, the allowable increase is 0.5, and allowable decrease is -big-M.

The parametric objective function is:

$$0U_1 + 2U_2 = 2 + 2C_1 - 2C_2$$

With shadow prices ($X_1 = 2, X_2 = -2$) which is solution to the primal with optimal value of $X_1 + 0X_2 = 2$, as expected.

9. SOFTWARE PACKAGES SHADOW PRICE AND SENSITIVITY ANALYSIS REPORTS

In the previous Sections we solved manually standard form problem P1 and the non-standard form problem P2 and then compared the results with the results obtained by some of the LP packages available in the market.

Unfortunately several of these educational LP software packages give misleading and incomplete results that could increase complexity for the manager. For instance, the widely used LINDO (2003) LP software does not provide any direct warnings about the existence of multiple nor degenerate optimal solutions. Or the other hand WinQSB (2003) might indicates “Note: Alternate Solution Exists!!” when in fact there might be none. As a result, the final report on sensitivity analysis might not be valid in these cases, Lin (2010). Similarly, several other LP software packages have their own limitations listed in Tables 6 and 7.

Table 6. Scope and limitations of some popular software

	Non-Negativity Conditions	Shadow Prices	Optimal Solution to non-standard form problems	Sensitivity Ranges
Lindo (2003)	Imposed on all variables by default, but can be changed to un-restricted	Non-consistent	May need conversion	May need conversion which is not an easy task
WinQSB (2003)	A variable must be either non-negative or unrestricted.	Consistent	May need conversion	Unreliable
QM for Windows (2003)	All variables must be non-negative	Non-consistent	May need conversion	Difficult to Convert
Management Scientist (1999)	All variables must be non-negative	Non-consistent	May need conversion	May need conversion which is not an easy task
Excel Solver (20013)	All variables are non-negative or un-restricted	Consistent	Readily Available	Reliable

Table 7. The standard form numerical example P1 summary results as a base for comparison with popular software

Variables Name	Optimal Solution	Objective Coefficient	Allowable Increase	Allowable Decrease
X1	375.0	18	M	6
X2	750.0	10	5	M
	Shadow	Slack	Allowable	Allowable
Constraints	Price	Surplus	Increase	Decrease
$12X_1 + 10X_2 \geq 12000$	2.125	0	6000	2000
$10X_1 + 15X_2 \leq 15000$	- 0.75	0	3000	5000

LINDO produces the output displayed in Listing 1 for the first numerical example P1, which is in standard form.

In this LINDO final report the dual prices are reported to be $U_1 = - 2.125$, and $U_2 = 0.75$, while we found earlier that the shadow prices are $U_1 = 2.125$, and $U_2 = - 0.75$. Even the term Dual Prices (i.e., Dual solution, are the shadow Prices) is confusing as the values $U_1 = - 2.125$, and $U_2 = 0.75$ are not a solution to the dual problem.

This misleading result is not limited to LINDO, unfortunately, at least two other LP software packages namely Quantitative Methods (QM 2003) and Management Scientist (1999) produce misleading results.

9.1 Shadow Prices Might Have Different Sign

When reading and interpreting Shadow Prices from software packages, managers must be especially attentive to the software description of the analysis.

There are no common accepted practices in Sensitivity Reports. For example, LINDO, Excel Solver and GAMS (2013) use different notation in their reports. In general, the terms shadow prices and dual prices are used interchangeable; however, if we compare the reports by LINDO and by hand-computation of numerical example P1,

Listing 1. Input and output of LINDO software package for the first numerical example

```
LINDO INPUT and OUTPUT

Minimize 18X1 +10X2
Subject to 12X1 + 10X2 ≥12000
10X1 +15X2 ≤ 15000
End

LP OPTIMUM FOUND AT STEP 2
      OBJECTIVE FUNCTION VALUE
    1) 14250.0
VARIABLE      VALUE      REDUCED COST
    X1         375.0         0.0
    X2         750.0         0.0
ROW   SLACK OR SURPLUS   DUAL PRICES
  2)           0.0         -2.125
  3)           0.0          0.75
NO. ITERATIONS= 2
RANGES IN WHICH THE BASIS IS UNCHANGED:
```

OBJ COEFFICIENT RANGES			
VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	18.0	INFINITY	6.0
X2	10.0	5.0	INFINITY

RIGHTHAND SIDE RANGES			
ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	12000.0	6000.0	2000.0
3	15000.0	3000.0	5000.0

we see that the dual prices from LINDO have the opposite sign of the shadow prices from Solver.

The Lindo (2013) Manual gives the following justification and interpretation:

...improve is a relative term. In a minimization problem, such as our example, interpreting the dual price requires some thought. The dual price ..., is-. This means raising the right-hand side of the... constraint by one unit would cause the objective to “improve” by negative 100. That is, it would increase ... In other words, the marginal

cost of providing one additional ...in additional salary expense.” “Dual prices are sometimes called shadow prices, because they tell you how much you should be willing to pay for additional units of a resource. Going back to the Friday constraint, if we use the shadow price interpretation of the dual price, we can also say that we should be willing to pay to have an additional... If ...available for less than .., then we might want to consider this option.” “As with reduced costs, dual prices are valid only over a limited range.

Notice that unfortunately Excel Manual (2013) also uses the terms like “unit change”, “unit increase” while defining shadow prices.

The dual value for a constraint is nonzero only when the constraint is equal to its bound. This is called a binding constraint, and its value was driven to the bound during the optimization process. Moving the constraint left hand side’s value away from the bound will worsen the objective function’s value; conversely, “loosening” the bound will improve the objective. The dual value measures the increase in the objective function’s value per unit increase in the constraint’s bound..... In the case of linear problems, the dual values remain constant over a range

Notice that the change by one unit may not be within the range, see sub-section 7.4.

Just to add to the confusion, other packages such as GAMS report marginal’s without any information about their range of validity. (see Listing 2)

Subtle discrepancies like this one are unfortunate. There is a pounding need for consensus

on the vocabulary and structure on the output of optimization software packages.

Using Excel LP Solver produces the output shown in Listing 3 for the non-standard form numerical example. (see Table 8)

The results are comparable with our hand computation. However one must be careful in any generalization, since this comparison is done on a specific numerical example.

WinQSB Software produces the following outputs for the non-standard form numerical example. The results with all constraints in explicit form, and then selecting the unrestricted option are shown in Listing 4.

The sensitivity ranges are not correct. They are differed from the results already obtained by hand computations.

9.2 Poor Scaling and Unstable Software Solutions

When modeling a business problem, different conditions might require us to use parameters or inputs having very different order of magnitudes. For example, an LP from manufacturing industry

Listing 2. Excerpt from GAMS output for the first numerical example that is in standard form

	LOWER	LEVEL	UPPER	MARGINAL
EQU constr1	12000.0	12000.0	+INF	2.1250
EQU constr2	-INF	15000.0	15000.0	-0.750

Table 8. Non-standard form numerical example P2 summary results

Variables Name	Optimal Solution	Objective Coefficient	Allowable Increase	Allowable Decrease
X1	2	1	M	1
X2	-2	0	0.5	M
Constraints Name	Shadow Price	Slack Surplus	Allowable Increase	Allowable Decrease
Constraint 1: $X1 + X2 \geq 0$	-1	0	1	M
Constraint 2: $2X1 + X2 \leq 2$	1	0	M	2

Listing 3. Excel result with all constraint in explicit form, and then selecting the unrestricted option

Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
X1	2	0	1	1E+30	1
X2	-2	0	0	0.5	1E+30
	Final	Shadow	Constraint	Allowable	Allowable
Name	Value	Price	R.H. Side	Increase	Decrease
Constraint 3: X1 >= 0	2	0	0	2	1E+30
Constraint 4: X2 <= 0	-2	0	0	1E+30	2
Constraint 1: X1 + X2 >= 0	0	-1	0	1	1E+30
Constraint 2: 2X1 + X2 <= 0	2	1	2	1E+30	2

Listing 4. WinQSB Input and Its Combined report

LP Input in Matrix Format

Variable	X1	X2	Direction	R.H.S.
Maximize	1	0		
Constraint 1	1	1	>=	0
Constraint 2	2	1	<=	2
Constraint 3	1	0	>=	0
Constraint 4	0	1	<=	0
Lower Bound	-M	-M		
Upper Bound	M	M		

Variable Type X1 Unrestricted, X2 Unrestricted

Decision Variable	Solution Value	Unit Profit Contribution	Allowable Min. c(j)	Allowable Max. c(j)
X1	2	1	0	1
X2	-2	0	-M	0

Objective Function (Max.) = 2.0

(Note: Alternate Solution Exists!!)

Constraint	Shadow Price	Allowable Min. RHS	Allowable Max. RHS
Constraint 1	-1	-M	1
Constraint 2	1	0	M
Constraint 3	2	-M	2
Constraint 4	2	-2	M

might include dollar amounts in millions and at the same time might include quality based conditions in the range of nanometers (one billionth of a meter). If an LP is modeled using the such units then the solvers such as Lindo and Excel gives us warning about poor scaling but then also gives some updated values. Frontline Solvers (the developers of Excel Solvers) admit to this limitation and the website reports that:

The effects of poor scaling in a large, complex optimization model can be among the most difficult problems to identify and resolve. It can cause Solver to return messages such as ‘Solver could not find a feasible solution’, ‘Solver could not improve the current solution,’ or even ‘The linearity conditions required by this LP Solver are not satisfied,’ with results that are suboptimal or otherwise very different from your expectations. (Frontline Solvers, 2013)

Unfortunately, Solver does not always alert the user about any issues with the problem. The following example clearly is prone to scaling issues with a very large coefficient for X_3 in the third constraint. (Depending on the computer a larger or smaller coefficient yields the issue describe here). However, when the problem is attempted with Solver, Solver sometimes yields an error message as the ones described earlier, but in most cases, it does not give any error message and reports as optimal an infeasible solution.

Numerical Example:

Minimize X_2

Subject to:

$$10X_1 - X_2 \leq 0$$

$$X_1 = 1$$

$$X_1 - 100000$$

$$X_3 \leq 0 \quad X_3 \leq 1$$

$$X_3 \text{ is positive integer}$$

Solver might report as the optimal solution $X_1 = 1$, $X_2 = 10$, and $X_3 = 0$. This solution violates constraint (3). The correct solution is $X_1 = 1$, $X_2 = 10$, and $X_3 = 1$. Therefore, we have numerical instability caused by too large values.

Excel recommends that to avoid such occurrences a model must only be allowed to vary by an order of magnitude of 3 (i.e. 1 to 1000). Excel allows the use of “Use Automatic Scaling” feature for automatically rescaling the values but it cannot rescale the problems “intelligently”. Hence if the automatic rescaling feature is used the outcomes could unknowingly become worse rather than better. Automatic rescaling can also slowly down the optimization process significantly (Winston and Albright, 2011). Similar limitations exist for Lindo also (Martin 1999) and may exist for several other solvers as well. However these limitations are largely and conveniently ignored.

One of the possible reasons for such ignorance could be due to the high expectations and the promise of convenience offered by LPs. Using LPs, a manager can get an optimal solution even when the problem includes a huge variety of factors (inputs and constraints) which could be coming from a wide variety of domains. For large and complex LPs, the managers and the coders would want to reduce the complexity associated with LPs and would prefer not to change the units for the sake of convenience. For example when developing LPs for manufacturing sector a manager (and even the coders) might prefer to use dollars values directly in the model instead of converting them to millions (which reduces the order of magnitude by six but then increase the chance of getting optimal dollar values in decimals) and use it with values of quality tolerance in a fraction of millimeter. Such instances are common and widespread. The authors lament that majority of books violates the ideal requirement of order of magnitude in the discussions and leaves the managers very much vulnerable to being unknowingly dependent on sub-optimal results.

The best approach recommended to deal with this situation is to use consistent scaling at the time of modeling. Managers should be aware of this issue and use consistent units for products and constraints. For example, if working with multiple monetary constraints they all should be scale in hundreds or thousands. Alternative if working with multiple products similar units should be used for all the products.

10. MAINTAINING A PRIMAL DEGENERATE OPTIMAL VERTEX OPTIMAL

Almost all research work in LP literature starts with an assumption of a non-degenerate primal/dual LP. Degeneracy is the case of having multiple vertices at the same point.

The necessary condition for the existence of LP degeneracy: If at optimal point the number of binding constraints is more than n (i.e., the dimension of the problem) then the solution to an LP might be degenerate.

Resolution: Double-check the coefficients of all constraints including the RHS values. There could have been rounding error.

10.1 False Degenerate Optimal Solution

Consider the following n=2 dimensional LP problem:

Numerical Example P2:

Maximize X_2

Subject to:

$$\begin{aligned} X_1 + X_2 &= 5 \\ -X_1 + X_2 &\leq 1 \\ X_1 &\geq 0, X_2 \geq 0 \end{aligned}$$

This problem has a unique, non-degenerate optimal solution at $(X_1 = 2, X_2 = 3)$. However, if one rewrites the equality constraint in the form of two inequalities, the equivalent problem is:

Maximize X_2

Subject to:

$$\begin{aligned} X_1 + X_2 &\leq 5 \\ X_1 + X_2 &\geq 5 \\ -X_1 + X_2 &\leq 1 \\ X_1 &\geq 0, X_2 \geq 0 \end{aligned}$$

Clearly, the optimal solution $(2, 3)$ now makes three constraints binding. Three is more than the dimension $n=2$ of this problem; therefore one may incorrectly conclude that this solution is a degenerate optimal solution. This simple numerical example serves the purpose of the necessary (but not a sufficient) condition for a degenerate optimal solution.

An occurrence of degeneracy can have a significant impact on the shadow prices and may cause oddities in the output provided by LP solvers. Moreover, the usual sensitivity analyses do not provide complete information in the degenerate case; that is, the information one obtains from most LP packages is a subset of the true sensitivity intervals. There are more effective approaches to cope with this problem; however, they are computationally much more involved, Lin (2010). Almost all software packages do not alert the user that the problem is degenerate.

Numerical Example P4:

Maximize $C(\mathbf{X}) = X_1 + X_2$

Subject to $X_1 \leq 1$

$$\begin{aligned} X_2 &\leq 1 \\ X_1 + X_2 &\leq 2 \\ X_1 &\geq 0 \\ X_2 &\geq 0 \end{aligned}$$

The optimal solution is $X_1 = 1$, and $X_2 = 1$, at which all three constraints are binding. We are interested in finding out how far the RHS of

binding constraints can change while maintaining the degenerate optimal solution. Because of degeneracy, there is dependency among these constraints at the binding position, that is:

$$\begin{aligned} X_1 &= 1 \\ X_2 &= 1 \\ X_1 + X_2 &= 2 \end{aligned}$$

In order to maintain the degenerate vertex while changing the RHS values, the RHS proportionately must be changed to the coefficients of the decision variables. The resulting parametric RHS system of equations is as follows:

$$\begin{aligned} X_1 &= 1 + 1r_1 + 0r_2 \\ X_2 &= 1 + 0r_1 + 1r_2 \\ X_1 + X_2 &= 2 + 1r_1 + 1r_2 \end{aligned}$$

The parameters on the RHS allow for the degenerate vertex to move maintaining its structure. Since there are two decision variables, any two of the parametric equations can be used to find the parametric degenerate optimal solution. For example, the first two equations provide $(X_1 = 1 + r_1, X_2 = 1 + r_2)$, which clearly satisfies the third one. As mentioned earlier, for larger problems one may use JavaScript software: <http://home.ubalt.edu/ntsbarsh/Business-stat/otherapplets/PaRHSSyEqu.htm>

The perturbed optimal value is $C(X) = X_1 + X_2 = 2 + r_1 + r_2$. In order for this vertex to remain optimal, it must satisfy all other constraints that have not been used. For this numerical example, the non-negativity conditions are the remaining constraints; therefore, we obtain the following conditions for r_1 and r_2 :

$$\{r_1 \text{ and } r_2 \mid r_1 \geq -1, r_2 \geq -1\}.$$

Notice that while the changes satisfy these conditions the degenerate solution remain degenerate, for any change outside of this set ensures the manager that the solution is not any longer

degenerate therefore the sensitivity and shadow prices are valid.

11. MAINTAINING THE MULTIPLE OPTIMAL VERTICES OPTIMAL

The necessary condition for the existence of LP multiple solutions: If in the “software final report” the total number of zeros in the Reduced Cost, together with number of zeros in the Shadow Price columns, exceeds the number of constraints, then you might have multiple solutions. Sensitivity analysis is not applicable. That is, the sensitivity analysis based on one optimal solution may not be valid for the others, Lin (2010). Software package like LINDO alert the user if a problem has multiple solutions, while others like Excel LP Solver do not.

Resolution: Check the coefficients in the objective function and the coefficients of the constraints. There could have been rounding error.

Numerical Example P5:

Consider the following LP:

$$\begin{aligned} \text{Maximize } C(X) &= 6X_1 + 4X_2 \\ \text{Subject to } X_1 + 2X_2 &\leq 16 \\ 3X_1 + 2X_2 &\leq 24 \\ X_1 &\geq 0 \\ X_2 &\geq 0 \end{aligned}$$

This LP has two optimal vertices; namely, $(X_1 = 8, X_2 = 0)$ and $(X_1 = 4, X_2 = 6)$. Notice that the existence of two solutions means we have innumerable optimal solutions. For example, for the above problem, for all parameter a , $0 \leq a \leq 1$, the following convex combinations are also optimal:

$$\begin{aligned} X_1 &= 8a + 4(1 - a) = 4 + 4a, \\ X_2 &= 0a + 6(1 - a) = 6 - 6a. \end{aligned}$$

The Dual LP is:

$$\begin{aligned} &\text{Minimize } 16U_1 + 24U_2 \\ &\text{Subject to } U_1 + 3U_2 \geq 6 \\ &\quad 2U_1 + 2U_2 \geq 4 \\ &\quad U_1 \geq 0 \\ &\quad U_2 \geq 0 \end{aligned}$$

This dual LP has a degenerate optimal solution ($U_1 = 0, U_2 = 2$). In order to maintain the degenerate vertex while changing the RHS, the RHS values must be changed in proportion to the coefficients of the decision variables. The parametric RHS system of equations is:

$$\begin{aligned} U_1 + 3U_2 &= 6 + 1c_1 + 3c_2 \\ 2U_1 + 2U_2 &= 4 + 2c_1 + 2c_2 \\ U_1 &= 0 + 1c_1 \end{aligned}$$

As before, any two of the parametric equations can be used to find the parametric degenerate optimal solution. For example, the first two equations provide $U_1 = c_1$ and $U_2 = 2 + c_2$, which clearly satisfies the third equation. For this vertex to remain optimal, it must satisfy all other constraints that have not been used. For this example, the non-negativity condition, $U_2 \geq 0$, is the only remaining constraint. Therefore, we obtain the following conditions for c_1 and c_2 :

$$\{c_1, c_2 \mid c_2 \geq -2\}.$$

Moreover, the perturbed cost coefficients $(6 + c_1)X_1 + (4 + c_2)X_2$ must be proportional to its parallel constraint $3X_1 + 2X_2 \leq 24$, i.e., $(6 + c_1)/3 = (4 + c_2)/2$. This simplifies to $2c_1 = 3c_2$. Putting together all these conditions, we obtain the largest sensitivity set as follows:

$$\{c_1, c_2 \mid c_2 \geq -2, 2c_1 = 3c_2\}.$$

Notice that, while the changes satisfy these conditions, the problem has multiple solutions; for any change outside of this set ensures the

manager that the problem does not any longer have multiple solutions therefore the sensitivity and shadow prices are valid.

12. THE MORE-FOR-LESS OR LESS-FOR-MORE SITUATIONS

A curious property called the more-for-less (MFL) or less-for-more (LFM) phenomenon is associated with some linear programs (LP). The existing literature has demonstrated the practicality and value of identifying cases where the paradoxical situation exists, Arsham (1996). For example, consider the following production LP.

Numerical Example P6

$$\begin{aligned} &\text{Maximize } C(X) = X_1 + 3X_2 + 2X_3 \\ &\text{Subject to } X_1 + 2X_2 + X_3 = 4 \\ &\quad 3X_1 + 2X_3 = 9 \\ &\quad X_1 \geq 0, X_2 \geq 0, X_3 \geq 0 \end{aligned}$$

The optimal solution for this LP occurs at vertex $X_1 = 1, X_2 = 0, X_3 = 3$, with the maximum value of objective function as \$7.

Note that, if the RHS of second constraint is increased to 12, then the new optimal solution gives the optimal value of $C(X) = \$4$; i.e., a decrease in profit while working more hours. This situation arises frequently in LP models and is known as the “more-for-less” paradox where further analysis could bring significant reduction in costs. We apply the primal sensitivity algorithm to solve the RHS perturbed equation constraints with $X_2 = 0$,

$$\begin{aligned} X_1 + X_3 &= 4 + r_1 \\ 3X_1 + 2X_3 &= 9 + r_2 \end{aligned}$$

The parametric solution is $X_1 = 1 - 2r_1 + r_2, X_2 = 0$, and $X_3 = 3 + 3r_1 - r_2$, with the optimal parametric solution as $C(X) = 7 + 4r_1 - r_2$. Plugging the parametric solution into other non-binding constraints, we obtain

$$\begin{aligned} X_1 &= 1 - 2r_1 + r_2 \geq 0 \\ X_3 &= 3 + 3r_1 - r_2 \geq 0 \end{aligned}$$

Looking at the parametric optimal function, notice that the shadow price of the second constraint, coefficient of r_2 , is negative. To find out the best number of hours, one must work to maximize the profit function $7 + 4r_1 - r_2$, by setting $r_1 = 0$ and finding the largest negative value for r_2 . Therefore, the constraints reduce to:

$$\begin{aligned} X_1 &= 1 + r_2 \geq 0 \\ X_3 &= 3 - r_2 \geq 0. \end{aligned}$$

The largest negative value is $r_2 = -1$. This gives the optimal solution of ($X_1 = 0, X_2 = 0, X_3 = 4$) with the optimal value of $C(\mathbf{X}) = 8$. Therefore, the optimal strategy is to work 8 hours instead of 9 hours.

Proposition 3: *A necessary and sufficient condition for the existence of a less-for-more (more-for-less) solution to a maximization (minimization) problem is the existence of an equality constraint with a negative shadow price.*

Theorem 1: *We are in LFM or MFL situation iff at least one of r_i 's in the above parametric LP formulation is negative [positive]*

13. CONCLUSION

A business environment is dynamic. A problem solution is valid in a limited time window only and is subject to revision in the next time window. Generally, a constraint set is less subject to change as compared to the objective function. For example, in production and transportation problems, the capacity constraints may remain rather stable over a period of time. On the other hand, profit coefficients of the objective function are inversely related to the price, which may fluctuate,

being determined by the market conditions and competition. The proposed method can be used to optimize LP problems with varying objective function. Given a system of linear equalities and/or inequalities, the method provides all vertices of the feasible region. By means of examples, we have illustrated the use of the Improved Algebraic Method to efficiently derive the slack and surplus amounts for the resources of an LP. The parametric representation quickly provides all dual prices to carry out analysis for desirability of obtaining additional resources. The parametric representation also allows one to study variability (i.e., uncertainties) of the coefficients of the objective functions and the right-hand-side of constraints.

We provide a comprehensive managerial coverage of linear programming post-optimality analysis. The collections of presented tools are easy to understand, easy to implement (for small size problems), and provide useful information to the manager. Therefore this collection could prove valuable to involve the manager throughout the decision-making process to understand therefore being implemented.

It is the proposed IAM approach to solve a system of inequalities that provides a bridge between the graphical method and the simplex method. The simplex method is an efficient computer implementation of algebraic methods and almost all the LP software use it.

All the LP tools are covered within the decision variable space; no additional variables such as slack/surplus/artificial variables or constraints are added.

The proposed approach provides useful information for the manager such as shadow prices and expands the sensitivity analysis scope by proving sensitivity regions rather than ranges that LP software provides. Under the proposed methods, the elaborate fundamental LP theorems fall naturally out as by-products. Moreover, it can also be used to fill the gap between the graphical method of solving LP problems and the simplex method.

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