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A TABULAR SIMPLEX-TYPE ALGORITHM AS A TEACHING AID FOR GENERAL LP MODELS

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TABLE OF CONTENTS

1. INTRODUCTION

The presentation of the simplex method, which solves linear programming (LP) problems, is not universal. In the U.S.A. instructors on the West coast enjoy solving the minimization problems, while in the East the maximization version is preferred. Even within each of these groups you will find differences in presenting the simplex rules.

Consider an LP problem in which some constraints are in (=) or ($\geqslant$) forms with the right-hand side (R.H.S.) non-negative. For this type of problem the usual simplex algorithm (big M-method) requires additional variables (artificial variables) and introducing penalty terms in the objective function. Enough experience has been gained showing that some students, particularly non-mathematical majors, have difficulty in understanding the intuitive notion of this algorithm. This is the reason and the key factor in motivating some authors in trying to develop algorithms which do not involve any artificial variables and penalty terms. Recently, several new algorithms which generally avoid the use of artificial variables, for the sake of simplicity, appeared in some textbooks (see, for example, Refs 1, p. 302; 2, p. 253] for such algorithms). Unfortunately, these algorithms are not for the general purpose of solving all types of LP problems and somehow mislead students. For example, in following all the steps given in these types of algorithms for the following problem:

$$\min_{X_1,X_2,X_3} X_1 + X_2 - X_3$$

$$\text{subject to } 2X_1 + X_2 - X_3 \geqslant 3$$

$$X_1, X_2, X_3 \geqslant 0$$

one never gets to the final tableau, in other words the algorithm never terminates. In Section 2 a new algorithm is presented which efficiently incorporates the regular and dual simplex algorithms. The strategy adapted for the new algorithm is summarized by the following two phases:

Phase I. Push toward a neighboring vertex of the optimal solution while trying to maintain feasibility.
Phase II. If pushed too far in Phase I, pull back toward the optimal vertex (if any).

2. A SUMMARY OF THE REFINED ALGORITHM

To start the algorithm the LP problem must be converted into the following standard form:

(a) Maximization problem.
(b) R.H.S. non-negative.
(c) All variables non-negative.

The following steps describe how the algorithm works along with the proof that it terminates successfully for any type of LP problem, provided there is no redundancy among the constraints:

Step 1. By introducing slack or surplus variables convert all inequalities (except non-negativity) constraints into equalities. The coefficient matrix must have full row rank.

Step 2. Construct the initial tableau containing all slack variables as basic variables. See the initial tableaux for the numerical examples given in the next section.

Step 3. Generate a basic variable (BV) set (not necessarily feasible) as follows:

(i) Incoming variable is X_j with largest C_j (the coefficient of X_j in the objective function). Do not consider the columns of the known basic variables. If there are alternatives choose any one (if this is the last variable to enter, choose the one having a non-negative column ratio, C/R, if any).

(ii) Is X_j the last variable to enter into BV set? If yes, enter it into the empty row and generate the next tableau (if impossible to pivot, go to Step 3(i) and choose the next C_j) and go to Step 4. Otherwise its row position is the row having the smallest positive C/R. Do not consider the rows which are already occupied. If there are alternative positions choose any one. Generate the next tableau and go to Step 3(i). If there is no positive C/R, then select the row having smallest absolute value C/R.

Proposition 1. By following Steps 3(i) and 3(ii) a complete BV set can always be generated which may not be feasible.

Proof. Proof of the first part of this statement follows by contradiction (from the fact that there is no redundancy/inconsistency among the constraints). The second part indicates that by pushing toward the optimal vertex we may have passed it.

Step 4. Are all $C_j \leqslant 0$ in the current tableau? If yes, go to Step 8; otherwise go to Step 5.

Step 5. Are all R.H.S. $\geqslant 0$ in the current tableau? If no, go to Step 7; otherwise go to Step 6.

Step 6. (i) Identify incoming variable (having largest C_j).
(ii) Identify outgoing variable (smallest non-negative C/R). If none, unbounded solution. (If more than one choose any one, this may be the sign of degeneracy.) Generate the next tableau and go to Step 4.

Step 7. Add the artificial constraint $\Sigma X_i + S = M$ to the table, with S as a basic variable, where M is an unspecified sufficiently large positive number, X_is are the current non-basic variables. Then enter X_j with the largest C_j in to the BV set and exit S, then generate the next tableau and go to Step 4.

Proposition 2. Following Step 7, all C_j become non-positive and remain as such.

Proof. Proof follows from a discussion given in Ref. [3, p. 265].

Step 8. Are all R.H.S. $\geqslant 0$? If yes, go to Step 10; otherwise go to Step 9.

Step 9. Identify the outgoing variable (smallest R.H.S.). Identify the incoming variable having negative coefficient in the pivot row in the current tableau, if there are alternatives choose the one with the smallest positive row ratio (R/R) (via row ratio test); if none, infeasible and stop; otherwise generate the next tableau and go to Step 8.

Step 10. This is the optimal solution, find out all multiple solutions if they exist (if the number of $C_j = 0$ is larger than the size of the BV set). If the solution is unbounded then at least one of the variables is a function of M. Stop.

Note that if all elements in any row are zero, we have one of two special cases. If the R.H.S. element is non-zero then the problem is infeasible. If the R.H.S. is zero this row represents a redundant constraint. Delete this row and proceed.

3. NUMERICAL EXAMPLES

The purpose of this section is to solve some very simple but diverse LP problems by walking through the proposed algorithm. These simple examples allow us to give a complete coverage of the algorithm while conserving space.

Example 1

Consider the following problem:

$$\min_{X_1, X_2} X_1 - 2X_2$$

$$\text{subject to } X_1 + X_2 \geqslant 2$$

$$X_1 - X_2 \leqslant -1$$

$$X_2 \leqslant 3$$

$$X_1, X_2 \geqslant 0.$$

The standard form of this problem is

$$\max_{X_1, X_2} -X_1 + 2X_2$$

$$\text{subject to } X_1 + X_2 \geqslant 2$$

$$-X_1 + X_2 \geqslant 1$$

$$X_2 \leqslant 3$$

$$X_1, X_2 \geqslant 0.$$

Converting the constraints into equality form, we have

$$X_1 + X_2 - S_1 = 2$$

$$-X_1 + X_2 - S_2 = 1$$

$$X_2 + S_3 = 3$$

$$X_1, X_2, S_1, S_2, S_3 \geqslant 0.$$

The initial tableau of this problem has the following form:

BV	X_1	X_2	S_1	S_2	S_3	R.H.S.	C/R
?	1	1	−1	0	0	2	2/1
?	−1	1	0	−1	0	1	1/1
S_3	0	1	0	0	1	3	
C_j	−1	2	0	0	0	0	

where the C_j row contains the coefficients of the X_j variables in the objective function. By Step 3(i) the incoming variable is X_2 and by Step 3(ii) its position is the second row. The next variable to enter is S_2.

BV	X_1	X_2	S_1	S_2	S_3	R.H.S.	C/R
S_2	2	0	−1	1	0	1	–
X_2	1	1	−1	0	0	2	–
S_3	−1	0	1	0	1	1	1
C_j	−3	0	2	0	0	−4	

A BV set is now completed. Step 4 directs us to Step 5 because at least one of the C_j is positive. Since the R.H.S. > 0, we go to Step 6 and by Step 6(i) the incoming variable is S_1 and by the C/R test S_3 must be replaced by S_1. Generating the next tableau, we obtain:

BV	X_1	X_2	S_1	S_2	S_3	R.H.S.
S_2	1	0	0	1	1	2
X_2	0	1	0	0	1	3
S_1	−1	0	1	0	1	1
C_j	−1	0	0	0	−2	−6

Now going back to Step 4, since $C_j \leqslant 0$, $\forall j$, we are directed to Step 8 and since all R.H.S. $\geqslant 0$, we go to Step 10 which states that this is an optimal solution: $X_1 = 0$, $X_2 = 3$ and optimal value $= -6$

Example 2

Consider the following problem which is infeasible:

$$\max_{X_1, X_2} X_1 - X_2$$

$$\text{subject to} \quad X_1 + 2X_2 \leqslant -2$$

$$X_1, X_2 \geqslant 0.$$

The standard form of the problem is

$$\max_{X_1, X_2} X_1 - X_2$$

$$\text{subject to} \quad -X_1 - 2X_2 \geqslant 2$$

$$X_1, X_2 \geqslant 0.$$

Introducing the surplus variable, the initial tableau has the following form:

BV	X_1	X_2	S_1	R.H.S.
?	−1	−2	−1	2
C_j	1	−1	0	0

The incoming variable is X_1 and after performing the row operation we have:

BV	X_1	X_2	S_1	R.H.S.
X_1	1	2	1	−2
C_j	0	−3	−1	2

Step 4 directs us to Step 8 and since R.H.S. $\leqslant 0$ the outgoing variable is X_1. To identify the incoming variable one must perform the R/R test:

R/R	0/1	−3/2	−1/1

Since none of the R/R is positive we conclude that the problem is infeasible.

Example 3

As another example consider the following problem:

$$\max_{X_1, X_2, X_3} \; -X_1 - X_2 + X_3$$

$$\text{subject to} \quad 2X_1 + X_2 - X_3 \geqslant 3$$

$$X_1, X_2, X_3 \geqslant 0.$$

The problem is already in stndard form, converting the inequality into an equality we get:

$$\max_{X_1, X_2, X_3} \; -X_1 - X_2 + X_3$$

$$\text{subject to } 2X_1 + X_2 - X_3 - S_1 = 3$$

$$X_1,\ X_2,\ X_3,\ S_1 \geqslant 0.$$

The initial tableau for this problem is:

BV	X_1	X_2	X_3	S_1	R.H.S.
?	2	1	−1	−1	3
C_j	−1	−1	1	0	0

The incoming variable is X_3. After performing the row operation, we have:

BV	X_1	X_2	X_3	S_1	R.H.S.
X_3	−2	−1	1	1	−3
C_j	1	0	0	−1	3

Step 4 directs us to Step 5, by which we are directed to Step 7 where we need to introduce the artificial constraint $X_1 + X_2 + S_1 + S = M$. Adding this constraint to the last tableau, we have:

BV	X_1	X_2	X_3	S_1	S	R.H.S.
X_3	−2	−1	1	1	0	−3
S	1	1	0	1	1	M
C_j	1	0	0	−1	0	3

By Step 7 the incoming variable is X_1 in place of S, generating the next tableau:

BV	X_1	X_2	X_3	S_1	S	R.H.S.
X_3	0	1	1	3	2	$2M - 3$
X_1	1	1	0	1	1	M
C_j	0	−1	0	−2	−1	$3 - M$

Going back to Step 4, all $C_j \leqslant 0$, going to Step 8, all R.H.S. $\geqslant 0$, then going to Step 10 we realize that this is indeed the final tableau with an unbounded solution. For this very simple LP problem the unboundedness is easily seen without any calculation.

Example 4

An LP problem with equality constraints:

$$\max_{X_i} \; X_1 + 3X_2 + 2X_3 + 2X_4$$

$$\text{subject to } X_1 + X_2 + X_3 = 2$$

$$X_2 + X_4 = 1$$

$$X_i \geqslant 0, \qquad i = 1, 2, 3, 4.$$

The initial tableau is:

BV	X_1	X_2	X_3	X_4	R.H.S.	C/R
?	1	1	1	0	2	2
?	0	1	0	1	1	1
C_j	1	3	2	2	0	

and the subsequent tableaux are as follows:

BV	X_1	X_2	X_3	X_4	R.H.S.
?	1	0	1	−1	1
X_2	0	1	0	1	1
C_j	1	0	2	−1	−3

BV	X_1	X_2	X_3	X_4	R.H.S.
X_3	1	0	1	−1	1
X_2	0	1	0	1	1
C_j	−1	0	0	1	−5

BV	X_1	X_2	X_3	X_4	R.H.S.
X_3	1	1	1	0	2
X_4	0	1	0	1	1
C_j	−1	−1	0	0	−6

The final tableau provides the optimal solution $X_1 = 0$, $X_2 = 0$, $X_3 = 2$, $X_4 = 1$ with optimal value = 6.

4. CONCLUDING REMARKS

We introduced an algorithm which has the following features:

—The algorithm terminates successfully for any type of LP problem.
—In comparison with other known general-purpose algorithms, the refined algorithm in tested examples generates the final tableau by, at most, an equal number of iterations.
—The refined algorithm is most suitable for LP problems with a large number of equality constraints, such as the classical transportation problem.
—The algorithm may require, at most, one artificial constraint.
—The developed algorithm does not involve any artificial variable nor any penalty term.
—Compared with other tabular methods of solving LP problems, the initial tableau and the subsequent tableaux generated by row operations look much simpler in the refined algorithm presentation. This is due to the fact that row operation is performed directly on C_js and there is no need to generate the reduced cost row known as C_j–Z_j.
—To reduce the number of iterations in this algorithm one can select the pivot column so as to maximize the total increase rather than the rate of increase. Such an idea has already been adopted in the LINDO package.

A further area of research includes the computer coding of the algorithm to study the computational behavior of the algorithm for large-scale problems and a computational comparison with other recent algorithms such as Karmarkar's and Khachian's.

At this point, the reader is asked to solve his/her own classroom LP problems by applying this algorithm as a final measure of evaluation.

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